Galois Covers

Report for the Seminar Galois Groups and Fundamental Groups

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Introduction

In this talk we will introduce the notion of a Galois cover. The classification of subcovers of a Galois cover is a topological analogon to the classification of subextensions of Galois extensions of fields. The main goal of this talk is to proof [Sza09, Theorem 2.2.10]. To highlight this analogy, we will compare [Sza09, Theorem 2.2.10] to its algebraic counterpart [Sza09, Theorem 1.3.11].

1 Basics About Connected Covers

In this chapter we give (a reminder of) the most important definitions and observations in the setting of connected covers.

Convention. The emptyset $\emptyset$ as a topological space is not connected.

Definition 1.1. For a cover $p : Y \to X$ we define

$$\mathrm{Aut}(p) = \{f : Y \to Y \mid f \text{ is a homeomorphism with } p \circ f = p\}$$

to be the *group of automorphisms of* p.

Remark. We note that by the definition of $\mathrm{Aut}(p)$, for each $x \in X$ we obtain a (left) group action $\mathrm{Aut}(p) \curvearrowright p^{-1}(x)$ by

$$\begin{aligned} - \cdot - \colon\ \mathrm{Aut}(p) \times p^{-1}(x) &\to p^{-1}(x) \\ (f, y) &\mapsto f(y) \end{aligned}$$

Remark. Let $p : Y \to X$ be a cover and $x \in X$.

Then by the definition of a cover there always exists an open neighborhood V of x such that there exists a family $(U_i)_{i \in I}$ of open subsets of Y fulfilling

- $p^{-1}(V) = \dot{\bigcup}_{i \in I} U_i$
- $p|_{U_i}$ restricts to a homeomorphism onto V for each $i \in I$.

We will call such a family $(U_i)_{i \in I}$ a *trivialization of* p *around* x.

Remark (Implications for Connected Covers)**.** For a possibly non-surjective connected cover $p : Y \to X$ (meaning X and Y are connected) one can show that p is surjective regardless. Furthermore one can proof that the cardinality of the fibre I is the same everywhere. Both statements actually only require connectedness of X.

Proposition 1.2 ([Sza09, Proposition 2.2.2])**.** Let $p : Y \to X$ be a cover, let Z be a connected topological space and $f, g : Z \to Y$ continuous maps with $p \circ f = p \circ g$. If there exists $z \in Z$ such that $f(z) = g(z)$, then $f = g$.

Proof. Let $z \in Z$ with $f(z) = g(z) = y$. Because p is a cover, we can pick a trivialization $(U_i)_{i \in I}$ of p around $p(y)$.

As $y \in \bigcup_{i \in I} U_i$, there exists a unique $i \in I$ with $y \in U_i$. Set $W = f^{-1}(U_i) \cap g^{-1}(U_i)$. Since $(p \circ f)|_W = p|_{U_i} \circ f|_W$ and $(p \circ g)_W = p|_{U_i} \circ g|_W$, we obtain $f|_W = g|_W$ from $p \circ f = p \circ g$ as $p|_{U_i}$ is a homeomorphism.

This shows that the set $A = \{z \in Z \mid f(z) = g(z)\}$ is open, because each element of A admits an open neighborhood that is contained A.

Now let $z' \in A^c$, which means $f(z') \neq g(z') = y'$. Because p is a cover, we obtain a trivialization $(U'_i)_{i \in I}$ around y'.

Assume there exists $j \in I$ such that $f(z') \in U'_j$ and $g(z') \in U'_j$. Because p restricted to U'_j is bijective, this implies $(p \circ f)(z') \neq (p \circ g)(z')$, which is a contradiction to our assumptions.

Thus for $j, k \in I$ with $f(z') \in U'_j$ and $f(z') \in U'_k$ by setting $W = f^{-1}(U'_j) \cap g^{-1}(U'_k)$ we have $f(w) \neq g(w)$ for all $w \in W$, which shows that A^c is also open.

Because A is not empty and Z is connected we conclude $Z = A$ which proves the proposition. ☐

Lemma 1.3 ([Sza09, Lemma 2.2.1])**.** Let $p : Y \to X$ be a connected cover and $\phi \in \mathrm{Aut}(p)$. If f has a fixed point, then $f = \mathrm{id}_Y$.

Proof. Apply Proposition 1.2 with $Z = Y$, $f = \mathrm{id}_Y$ and $g = \phi$. ☐

2 Galois Covers

In the following we introduce the notion of Galois covers characterize them as the covers with automorphism group acting transitively on the fibres.

Proposition 2.1 ([Sza09, Proposition 2.2.3])**.** Let $p : Y \to X$ be a connected cover. Then the action $\mathrm{Aut}(p) \curvearrowright Y$ is even.

Proof. For $y \in Y$ choose a trivialization $(U_i)_{i \in I}$ around $p(y)$. Let $i \in I$ such that $y \in U_i$ and choose $f \in \mathrm{Aut}(p)$ with $f(U_i) \cap U_i \neq \emptyset$.

For $x \in f(U_i) \cap U_i$ we have $f^{-1}(x) \in U_i$ as well as $x \in U_i$. Because $(p \circ f^{-1})(x) = p(x)$ and p restricted to U_i is bijective, we have $f^{-1}(x) = x$. By Lemma 1.3 we thus have $f^{-1} = \mathrm{id}_Y$.

This shows that U_i is an open neighborhood of y such that for $f \in \mathrm{Aut}(p)$ with $f(U_i) \cap U_i \neq \emptyset$ we have $f = \mathrm{id}_Y$, thus $\mathrm{Aut}(p) \curvearrowright Y$ is even. ☐

Proposition 2.2 ([Sza09, Proposition 2.2.4])**.** Let G be a group acting evenly on a connected space Y. Then

$$\begin{aligned} F \colon G &\to \mathrm{Aut}(p_G) \\ g &\mapsto (y \mapsto g \cdot y) \end{aligned}$$

is a group isomorphism, where $p_G : Y \to G \setminus Y$ is the canonical projection.

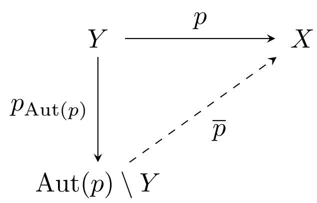
Proof. By [Sza09, Lemma 2.1.7] we know that p_G is indeed a cover.

It is clear that F is a well defined group homomorphism, as $p \circ F(g) = p$ holds by definition of $G \setminus Y$. Thus it suffices to show bijectivity.

For injectivity, let $g \in G$ with $F(g) = \mathrm{id}_Y$. Because the action of G is even, it is free and thus $g = e$.

For surjectivity pick $\phi \in \mathrm{Aut}(p_G)$ and let $y \in Y$ be arbitrary. As $(p \circ \phi)(y) = p(y)$, $\phi(y)$ is in the orbit of y by the action of G and thus we can find a $g \in G$ with $F(g)(y) = \phi(y)$. Then by Lemma 1.3 we have $\phi = F(g)$, which finishes the proof. □

Remark. Provided a connected cover $p : Y \to X$, we can form the quotient by the even (see Proposition 2.1) action $\mathrm{Aut}(p) \curvearrowright Y$. Let $[y] \in \mathrm{Aut}(p) \setminus Y$ be the orbit of $y \in Y$. For $y_1, y_2 \in [y]$ there is an $f \in \mathrm{Aut}(p)$ with $y_1 = f(y_2)$ and thus $p(y_1) = (p \circ f)(y_2) = p(y_2)$. Therefore, by the universal property of the quotient $\mathrm{Aut}(p) \setminus Y$ we obtain a unique continuous map $\overline{p} : \mathrm{Aut}(p) \setminus Y \to X$ such that



commutes. By Proposition 2.2 we have $\mathrm{Aut}(p_{\mathrm{Aut}(p)}) \cong \mathrm{Aut}(p)$.

Definition 2.3. A cover $p : Y \to X$ is *Galois* if it is connected and the induced map $\overline{p}\colon \mathrm{Aut}(p) \setminus Y \to X$ of the preceding remark is a homeomorphism.

Example 2.4. In the setting of Proposition 2.2 the map p_G is a Galois cover as $\mathrm{Aut}(p_G) = G$ and $\overline{p}_G = \mathrm{id}_{G \setminus Y}$.

Proposition 2.5 ([Sza09, Proposition 2.2.7])**.** A connected cover $p : Y \to X$ is Galois if and only if $\mathrm{Aut}(p)$ acts transitively on all fibres of p.

Proof. Acting transitively on a fibre $p^{-1}(x)$ is equivalent to statement that the orbit of any $y \in p^{-1}(x)$ is the whole fibre. This is equivalent to $\overline{p}$ being bijective by the definition of $\mathrm{Aut}(p) \setminus Y$. Therefore if $\overline{p}$ is a homeomorphism (and therefore bijective) p is Galois.

For the other implication, it suffices to show that $\overline{p}$ is an open map, as a continuous map that is open and and bijective is a homeomorphism. Let $U \subset \mathrm{Aut}(p) \setminus Y$ be open. Then $\overline{p}(U) = p(p_{\mathrm{Aut}(p)}^{-1}(U))$ and because p is a cover and thus open, $\overline{p}(U)$ is open and the proposition follows. □

3 Classification of Connected Subcovers

In this last section we will prove some preliminary lemmas used in the main result of this talk, the classification of connected subcovers of a Galois cover. Finally, we will compare this result to its algebraic counterpart.

Lemma 3.1 ([Sza09, Lemma 2.2.11])**.** Let $q : Z \to X$ be a connected cover with X locally connected and let $f : Y \to Z$ be a continuous map such that $q \circ f$ is also a cover. Then f is a cover.

Proof. Let $z \in \operatorname{im} f$, $x = q(z)$ and V be a connected open set that admits both a trivialization $(U_i)_{i \in I}$ of $p = q \circ f$ and a trivialization $(V_j)_{j \in J}$ of q around x. Such a set V exists because p and q are covers and X is locally connected.

We have decompositions $p^{-1}(V) = \dot{\bigcup}_{i \in I} U_i$ and $q^{-1}(V) = \dot{\bigcup}_{j \in J} V_j$ and because U_i is connected (as it is homeomorphic to V), $f(U_i)$ is also connected. Let $j \in J$ be the unique index such that $z \in V_j$. Because $z \in \operatorname{im} f$ we know $f^{-1}(V_j)$ is nonempty.

Pick an $i \in I$ with $f^{-1}(V_j) \cap U_i \neq \emptyset$. Due to $f(U_i) \subset \dot{\bigcup}_{j \in J} V_j$ and the fact that $f(U_i)$ is connected, $f(U_i) \subset V_j$ as it is the connected component containing z. Now $q|_{V_j} \circ f|_{U_i} = p|_{U_i}$ and thus $f|_{U_i} = q|_{V_j}^{-1} \circ p|_{U_i}$ which shows that $f|_{U_i}$ is a homeomorphism onto its image V_j.

From this and $f^{-1}(V_j) \subset p^{-1}(V)$ we obtain that $f^{-1}(V_j) = \dot{\bigcup}_{\{i \in I | f(U_i) = V_j\}} U_i$ is a trivialization around $z \in Z$.

It thus remains to show that f is surjective.

From what we have already proven we observe that $\operatorname{im} f$ is open. For $z \notin \operatorname{im} f$ the argument above shows that $\operatorname{im} f \cap V_j = \emptyset$, where V_j is again the component of $q^{-1}(V)$ containing z (otherwise $V_j \subset \operatorname{im} f$). Thus $(\operatorname{im} f)^c$ is open as well, which implies $\operatorname{im} f = Z$ by connectedness of Z.

This finishes the proof. □

Lemma 3.2. Let $q : Y \to Z$ be a Galois cover with X and Z locally connected and let $f : Z \to X$ be a continuous map such that $f \circ q$ is a cover. Then f is a cover.

Proof. For $x \in X$ choose an open connected neigborhood V that admits a trivialization $(U_i)_{i \in I}$ of p around x. Let $f^{-1}(V) = \dot{\bigcup}_{j \in J} V_j$ where the V_j are the connected components of $f^{-1}(V)$. These are open because Z is locally connected.

By the same connectedness arguments as in Lemma 3.1, we know that for every $i \in I$ there exists a unique $j \in J$ such that $q(U_i) \subset V_j$. We want to show that $q(U_i) = V_j$, because then we can pick a subset $I' \subset I$ such that $(q(U_i))_{i \in I'}$ is a disjoint family.

Let $z \in V_j$. Due to $f^{-1}(V) = q\left(\dot{\bigcup}_{i \in I} U_i\right)$ there is a $k \in I$ and $y \in U_k$ such that $q(y) \in V_j$, which implies $q(U_k) \subset V_j$. If there is $x \in q(U_i) \cap q(U_k)$, we can pick $\phi \in \operatorname{Aut}(q) \subset \operatorname{Aut}(p)$ such that $U_k = \phi(U_i)$. We then have $\phi(U_i) \cap U_i \neq \emptyset$ which implies $\phi = \operatorname{id}$ and thus $k = i$. Therefore V_j can be decomposed as a disjoint union of open sets $(p(U_l))_{l \in L}$ with $L \subset I$ and $i \in L$, which by connectedness of V_j implies that $p(U_i) = V_j$.

Now we can obtain a trivialization of V_j by picking $I' \subset I$ such that $(q(U_i))_{i \in I'}$. As $q|_{U_i} : U_i \to V_j$ is bijective and thus a homeomorphism, $f|_{V_j}$ must be a homeomorphism because $f|_{V_j} \circ q|_{U_i} = p|_{U_i}$.

This finishes the proof. □

Remark. In the situation of Proposition 2.5, it suffices for $\operatorname{Aut}(p)$ to act transitively on a single fibre. Because $p_{\operatorname{Aut}(p)} : Y \to \operatorname{Aut}(p) \setminus Y$ is Galois, by Lemma 3.2 the map

$\overline{p}$ is a cover. Then the fibre of the induced map $\overline{p}$ is a one point set everywhere (as $\overline{p}$ is a connected cover, it suffices to know the fibre at a single point). This again shows that $\overline{p}$ is bijective.

Definition 3.3. For a cover $p : Y \to X$ let $\mathrm{Subcov}(p)$ be the category whose

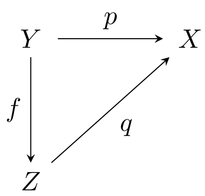
- objects are pairs (q, f) with $q \colon Z \to X$ a connected cover and $f \colon Y \to Z$ a continuous map with $q \circ f = p$
- morphisms between objects (q_1, f_1), (q_2, f_2) are continuous maps $g : Z_1 \to Z_2$ such that $q_2 \circ g = q_1$ and $g \circ f_1 = f_2$.

Remark. In Definition 3.3, from $q \circ f = p$ and Lemma 3.1 we have that f is a cover. Similarly, a morphism $g : Z_1 \to Z_2$ that is a morphism from (q_1, f_1) to (q_2, f_2) is also cover. Thus all morphisms occuring in Definition 3.3 are covers.

The next theorem is the main part of the talk.

Proposition 3.4 ([Sza09, Essentially Theorem 2.2.10])**.** Let $p \colon Y \to X$ be a Galois cover with X locally connected. Then

- for each subgroup $H \subset \mathrm{Aut}(p) = G$, the map $\overline{p}_H \colon H \setminus Y \to X$ is a connected cover.
- if $q : Z \to X$ is a connected cover such that



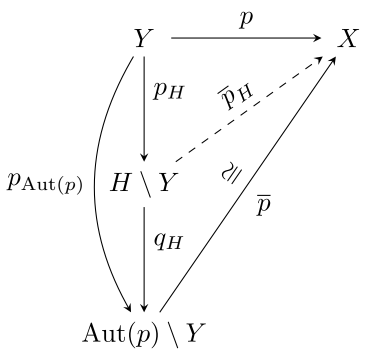
commutes for a continuous map $f \colon Y \to Z$, then f is a Galois cover and $q \cong \overline{p}_H$ in $\mathrm{Subcov}(p)$ for $H = \mathrm{Aut}(f) \subset \mathrm{Aut}(p) = G$. The cover q is Galois if and only if H is a normal subgroup of G and in this case we have $\mathrm{Aut}(q) \cong {}^{G}\!/_{H}$.

Furthermore, the above gives mutually inverse equivalences of categories

$$\begin{array}{rcl} \{\text{Subgroups of } G\} & \longleftrightarrow & \mathrm{Subcov}(p) \\ H & \longmapsto & (\overline{p}_H, p_H) \\ \mathrm{Aut}(f) & \longleftarrow\!\shortmid & (q, f) \end{array}$$

where we view $\{\text{Subgroups of } G\}$ as a category via the partial ordering induced by inclusions of subgroups of G.

Proof. We have the following commutative diagram illustrating the most important maps:



We first show that for $H \subset G$ the map $\overline{p}_H : H \setminus Y \to X$ is a cover. We know that p_H is a cover due to [Sza09, Lemma 2.1.7] as H acts evenly on Y. Because H acts transitively on the fibres of p_H by construction, p_H is Galois (see Example 2.4). Thus because $\overline{p}_H \circ p_H = p$ we have that $\overline{p}_H$ is a cover from Lemma 3.2.

Next we proof that a continuous map f with $q \circ f = p$ is a Galois cover. From Lemma 3.1 it follows that f is a cover. By Proposition 2.5 it thus suffices to check transitivity of the action of $H = \mathrm{Aut}(f)$ on fibre of f. Let $z \in Z$ and $y_1, y_2 \in f^{-1}(z)$. We have $y_1, y_2 \in p^{-1}(q(z))$ and because p is Galois there is an $g \in G$ with $g(y_1) = y_2$. We have to show that $g \in H$. As $(f \circ g)(y_1) = f(y_1)$ and $q \circ f \circ g = q \circ f$, Proposition 1.2 shows that $f \circ g = f$ and thus $g \in H$.

Next we show that $(q, f) \cong (\overline{p}_H, p_H)$ for $H = \mathrm{Aut}(f)$ induced by a map $f : Y \to Z$ with $q \circ f = p$ (which we know to be a Galois cover). From the universal property of quotients we obtain a map $\overline{f} : H \setminus Y \to Z$ which is a homeomorphism as f is Galois. Because $p = q \circ \overline{f} \circ p_H$ and $p = \overline{p}_H \circ p_H$ we obtain from surjectivity of p_H that $q \circ \overline{f} = \overline{p}_H$, proving $(q, f) \cong (\overline{p}_H, p_H)$.

We now proof that for (q, f), q is Galois if and only if $\mathrm{Aut}(f) = H \subset G$ is normal. First, let q be Galois. Our goal is the construct a group homomorphism $A : \mathrm{Aut}(p) \to \mathrm{Aut}(q)$ mit $\ker A = H$, which shows H is normal. Let $\phi \in \mathrm{Aut}(p)$ and $y \in Y$, then $f(y), f(\phi(y)) \in q^{-1}(f(y))$. By transitivity of q on its fibres there exists a $\widetilde{\phi} \in \mathrm{Aut}(q)$ with $\widetilde{\phi}(f(y)) = f(\phi(y))$. Therefore by Proposition 1.2 for each $\phi \in \mathrm{Aut}(p)$ there is a unique $\widetilde{\phi} \in \mathrm{Aut}(q)$ with $\widetilde{\phi} \circ f = f \circ \phi$. This construction gives us a map $A : \mathrm{Aut}(p) \to \mathrm{Aut}(q)$. We now show that this is a group homomorphism. We have $A(\phi_1 \cdot \phi_2)(f(y)) = \widetilde{\phi_1 \cdot \phi_2}(f(y)) = f((\phi_1 \cdot \phi_2)(y)) = \widetilde{\phi_1}(f(\phi_2)(y)) = \widetilde{\phi_1}(\widetilde{\phi_2}(f(y))) = A(\phi_1) \cdot A(\phi_2)$ and thus $A(\phi_1 \cdot \phi_2) = A(\phi_1) \cdot A(\phi_2)$.

It remains to show that $\ker A = H$. For $\phi \in \ker A$, we know $f = f \circ \phi$ and thus $f \in \mathrm{Aut}(f)$. Conversly for $\tau \in H$ by uniqueness we have $A(\tau) = \mathrm{id}_Y$. Therefore if q is Galois then H is normal and $\mathrm{Aut}(q) \cong {}^{\mathrm{Aut}(p)}\!/_{\mathrm{Aut}(f)}$.

Next we show that if $\mathrm{Aut}(f) = H \subset G$ is normal, then q is Galois.

The action $G \curvearrowright Y$ induces an action ${}^{G}\!/_{H} \curvearrowright H \setminus Y$. That this action is even can be seen by projecting an open $U \subset Y$ with $g \cdot U \cap U = \emptyset$ for all $g \in G \setminus \{e\}$ onto $H \setminus Y$

via p_H. We have a group isomorphism

$$\begin{aligned} \operatorname{Aut}(\overline{p}_H) &\to \operatorname{Aut}(q) \\ g &\mapsto \overline{f} \circ g \circ \overline{f}^{-1} \end{aligned}$$

because $\overline{f}$ is a homeomorphism fulfilling $\overline{f} \circ p_H = f$ and $q \circ \overline{f} = \overline{p}_H$. As ${}^{G}\!/_{H} \backslash (H \setminus Y) \cong X$ via the projection induced by $\overline{p}_H$, we know that the canonical map $\operatorname{Aut}(q) \setminus Z \to X$ is bijective and thus also a homeomorphism as it is open. Therefore q is Galois.

We now show functioriality of

$$\begin{aligned} F : \{\text{Subgroups of } G\} &\to \operatorname{Subcov}(p) \\ H &\mapsto (\overline{p}_H, p_H) \ . \end{aligned}$$

By construction it is clear that $F(\mathrm{id}_H) = \mathrm{id}_{(\overline{p}_H, p_H)}$. Thus it suffices to proof compatibility with inclusions. For $H \subset K \subset L$ subgroups in G, we have that $F(H \subset K) \circ F(K \subset L) = (H \setminus Y \to K \setminus Y) \circ (K \setminus Y \to L \setminus Y) = (H \setminus Y \to L \setminus Y) = F(H \subset L)$ as the maps are projections. This proves functioriality of F.

Next we show the functioriality of

$$\begin{aligned} G : \operatorname{Subcov}(p) &\to \{\text{Subgroups of } G\} \\ (q, f) &\mapsto \operatorname{Aut}(f) \ . \end{aligned}$$

Here we also have $G(\mathrm{id}_{(q,f)}) = \mathrm{id}_{\operatorname{Aut}(f)}$ by construction. We first show that a map $g : (q_1, f_1) \to (q_2, f_2)$ induces an inclusion $\operatorname{Aut}(f_1) \subset \operatorname{Aut}(f_2)$.

Let $\phi \in \operatorname{Aut}(f_1)$. Then $f_1 \circ \phi = f_1$ implies $g \circ f_1 \circ \phi = g \circ f_1$, which by $g \circ f_1 = f_2$ proves $\phi \in \operatorname{Aut}(f_2)$. Thus for maps $g_1 : (q_1, f_1) \to (q_2, f_2)$ and $g_2 : (q_2, f_2) \to (q_3, f_3)$ we have $\operatorname{Aut}(f_1) \subset \operatorname{Aut}(f_2) \subset \operatorname{Aut}(f_3)$ which proves functioriality of G.

Lastly we show that F and G are mutually inverse equivalences of categories. From what we have already proven, we know that $G \circ F = \mathrm{id}_{\{\text{Subgroups of } G\}}$.

Thus we only need to show $G \circ F \simeq \mathrm{id}_{\operatorname{Subcov}(p)}$. For $g : Z_1 \to Z_2$ a map from (q_1, f_1) to (q_2, f_2) the square

$$\begin{array}{ccc} (\overline{p}_{\operatorname{Aut}(f_1)}, p_{\operatorname{Aut}(f_1)}) & \xrightarrow{\ \overline{f}_1\ } & (q_1, f_1) \\ \big\downarrow & & \big\downarrow g \\ (\overline{p}_{\operatorname{Aut}(f_2)}, p_{\operatorname{Aut}(f_2)}) & \xrightarrow[\ \overline{f}_2\ ]{} & (q_2, f_2) \end{array}$$

where

- $\overline{f}_1 : \operatorname{Aut}(f_1) \setminus Y \to Z_1$ and $\overline{f}_2 : \operatorname{Aut}(f_2) \setminus Y \to Z_2$ are homeomorphisms as f_1 and f_2 are Galois

- the left vertical map $\mathrm{Aut}(f_1) \setminus Y \to \mathrm{Aut}(f_2) \setminus Y$ is the projection induced by $\mathrm{Aut}(f_1) \subset \mathrm{Aut}(f_2)$

commutes. As the horizontal maps are isomorphisms in $\mathrm{Subcov}(p)$ we thus have proven $G \circ F \simeq \mathrm{id}_{\mathrm{Subcov}(p)}$. □

We now state the algebraic counterpart to our main result Proposition 3.4 as in [Sza09, Theorem 1.3.11].

Theorem ([Sza09, Theorem 1.3.11] (Krull))**.** Let L be a subextension of the Galois extension $K|k$. Then $\mathrm{Gal}(K|L)$ is a closed subgroup of $\mathrm{Gal}(K|k)$. Moreover, the maps

$$L \mapsto H := \mathrm{Gal}(K|L) \quad \text{and} \quad H \mapsto L := K^H$$

yield an inclusion-reversing bijection between subfields $K \supset L \supset k$ and closed subgroups $H \subset G$. A subextension $L|k$ is Galois over k if and only if $\mathrm{Gal}(K|L)$ is normal in $\mathrm{Gal}(K|k)$; in this case there is a natural isomorphism $\mathrm{Gal}(L|k) \cong {\mathrm{Gal}(K|k)}/{\mathrm{Gal}(K|L)}$.

Remark (Comparison of Proposition 3.4 to [Sza09, Theorem 1.3.11])**.** In the following we will highlight the similarity between the aforementioned theorems. We use the same notions as in the respective theorems.

Cover Theory	Field Extensions
Galois cover $p : Y \to X$	Galois extension $K\|k$
$(q, f) \in \mathrm{Subcov}(p)$	Subextension $K \supset L \supset k$
$(q, f) \mapsto \mathrm{Aut}(f)$	$L \mapsto \mathrm{Gal}(K\|L)$
$H \mapsto (\overline{p}_H, p_H)$	$H \mapsto K^H$
q Galois iff $\mathrm{Aut}(f) \subset \mathrm{Aut}(p)$ normal	$L\|k$ Galois iff $\mathrm{Gal}(K\|L) \subset \mathrm{Gal}(K\|k)$ normal
For q Galois $\mathrm{Aut}(q) \cong {\mathrm{Aut}(p)}/{\mathrm{Aut}(f)}$	For $L\|k$ Galois $\mathrm{Gal}(L\|k) \cong {\mathrm{Gal}(K\|k)}/{\mathrm{Gal}(K\|L)}$

Outlook. This correspondence will be made precise in the later parts of the seminar. The bridge will be provided by Riemann surfaces.

References

[Löh18] Prof. Dr. Clara Löh. *Algebraic Topology.* An introductory course. 2018. URL: http://www.mathematik.uni-regensburg.de/loeh/teaching/topologie1_ws1819/lecture_notes.pdf (visited on 04/01/2021).

[Sza09] Tamás Szamuely. *Galois Groups and Fundamental Groups.* Cambridge Studies in Advanced Mathematics. Cambridge University Press, 2009. DOI: 10.1017/CBO9780511627064.