Seminar:
Galoisgroups and Fundamentalgroups

Talk 9 - Riemann surfaces

Preface
The following talk is devided into two parts. The first part defines the basic terminology and some examples. The second part deals with holomorphic maps between Riemann surfaces from a topological viewpoint.

Part I.

Definition 3.1.1 (complex atlas)
Let X be a Hausdorff topological space. A complex atlas on X is an open covering $\mathcal{U} = \{U_i \mid i \in I\}$ of X together with maps $f_i : U_i \to \mathbb{C}$ mapping U_i homeomorphically onto an open subset of $\mathbb{C}$ such that for all $i, j \in I$ the transition maps

$$f_j \circ f_i^{-1} : f_i(U_i \cap U_j) \to \mathbb{C}$$
$$f_i \circ f_j^{-1} : f_j(U_i \cap U_j) \to \mathbb{C}$$

are holomorphic.
The maps f_i are called complex charts. Two complex atlases $\mathcal{U} = \{U_i \mid i \in I\}$ and $\mathcal{U}' = \{U_i' \mid i \in I'\}$ on X are equivalent if their Union is also a complex atlas and the maps $f_j' \circ f_i^{-1} : f_i(U_i \cap U_j') \to \mathbb{C}$ are holomorphic for all $U_i \in \mathcal{U}$ and $U_j \in \mathcal{U}'$.

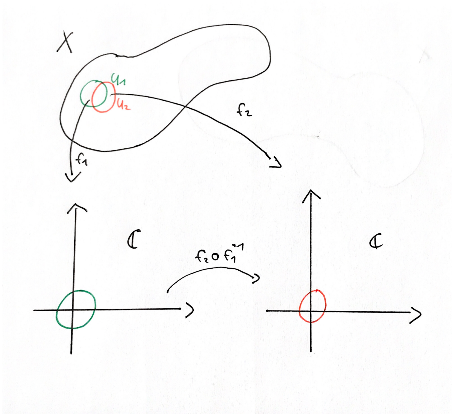

Abbildung 1: Topological building blocks of an complex manifold X

Definition 3.1.2 (Riemann surface)
A Riemann surface (or 1-dimensional complex manifold) is a Hausdorff space together with an equivalence class of complex atlases. We call the equivalence class of atlases occuring in the Definition 3.1.1 as the complex structure on the Riemann surface.

Example 3.1.3

1) Open subsets
Let $U \subset \mathbb{C}$ be a open subset. Open subset are endowed with a structure of a Riemann surface by the trivial covering $\mathcal{U} = U$ and the inclusion $i : U \to \mathbb{C}$.

2) The complex projective line
In order to construct the complex projective line we first extend the complex plane $\mathbb{C}$ with a point at infinity, so we get: $\mathcal{P}^1(\mathbb{C}) := \mathbb{C} \cup \{\infty\}$. The topology on $\mathcal{P}^1(\mathbb{C})$ is called the one-point compactification or Alexandroff compactification, which we know from complex analysis. The Alexandroff compactification is defined as: A subset $U \subset \mathcal{P}^1(\mathbb{C})$ is open $\Leftrightarrow$ $U \subset \mathbb{C}$ is open or $\infty \in U$ and $\mathcal{P}^1 \setminus U \subset \mathbb{C}$ is compact . By defining the stereographic projection (its a basic fact from analysis on manifolds that the stereographic projection is indeed an homeomorphism):

$$P : \mathbb{C} \cup \{\infty\} \to \mathbb{S}^2$$
$$z \mapsto \sigma(z) := \left(\frac{2z}{|z|^2+1}, \frac{|z|^2-1}{|z|^2+1}\right) \in \mathbb{C} \times \mathbb{R}$$

and identifying $z \in \mathbb{C} \cup \{\infty\}$ with $\sigma(z) \in \mathbb{S}^2$ we get that $\mathcal{P}^1(\mathbb{C}) \cong \mathbb{S}^2$

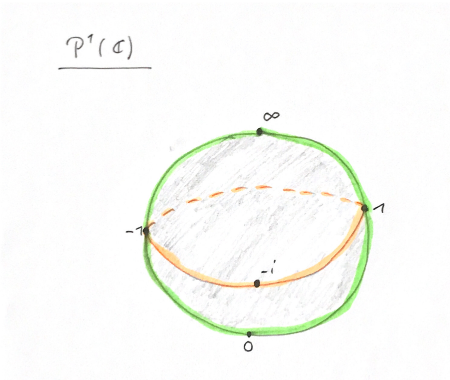

Abbildung 2: Complex projective line

Now we define two charts as follows:

$$f_0 : \mathcal{P}^1 \setminus \{\infty\} = \mathbb{C} \to \mathbb{C}$$
$$z \mapsto z$$
$$f_\infty : \mathcal{P}^1 \setminus \{0\} \to \mathbb{C}$$
$$z \mapsto \tfrac{1}{z} \text{ if } z \in \mathbb{C}$$
$$z \mapsto 0 \text{ if } z = \infty$$

The transition map is given as:

$$f_{0\infty} : f_0(\mathcal{P}^1 \setminus \{0, \infty\}) = \mathbb{C} \setminus \{0\} \to f_\infty(\mathcal{P}^1 \setminus \{0, \infty\}) = \mathbb{C} \setminus \{0\}$$
$$z \mapsto \tfrac{1}{z}$$

Since $z \mapsto \frac{1}{z}$ is holomorphic on $\mathbb{C} \setminus \{0\}$ we get an complex atlas $\{f_0, f_\infty\}$

Complex tori

To construct the complex tori we first need to clear the term lattice in mathematical terms. A lattice Γ is a discrete subgroup of $(\mathbb{C}, +)$ which is isomorphic to $\mathbb{Z}^2$ and spans $\mathbb{C}$ as an $\mathbb{R}-$vectorspace, explicitly in our case we set:

$$\begin{aligned} &\mathbb{Z}^2 \to \Gamma \subset \mathbb{C} \\ &(1,0) \mapsto w_1 \\ &(0,1) \mapsto w_2 \end{aligned}$$

w_1, w_2 are our basis of $\mathbb{C}$ over $\mathbb{R}$ with $\Gamma := \{n \cdot w_1 + m \cdot w_2 \mid n, m \in \mathbb{Z}\} \subset \mathbb{C}$. We construct on the quotientgroup $T := \mathbb{C}/\Gamma$ the complex structure as follows: Let $\pi : \mathbb{C} \to \mathbb{C}/\Gamma$ be the projectionmap. We now view T as an topological space with the quotient topology, which means a subset $U \subset T$ is open $\Leftrightarrow \pi^{-1}(U) \subset \mathbb{C}$ is open. Since $\mathbb{C}$ is connected, T is also connected. Also T is compact, it is covered by the image under the projection π of the compact parallelogramm which we denote by: $P := \{\lambda \cdot w_1 + \nu \cdot w_2 \mid \lambda, \nu \in [0,1]\}$. Defining the homeomorphism:

$$\begin{aligned} &f : T \to \mathbb{S}^1 \times \mathbb{S}^1 \subset \mathbb{C}^2 \\ &(\lambda \cdot w_1 + \nu \cdot w_2) + \Gamma \mapsto \left(e^{2\pi i \lambda}, e^{2\pi i \nu}\right) \end{aligned}$$

It follows that $T \cong \mathbb{S}^1 \times \mathbb{S}^1$. We can now construct the charts on T by setting

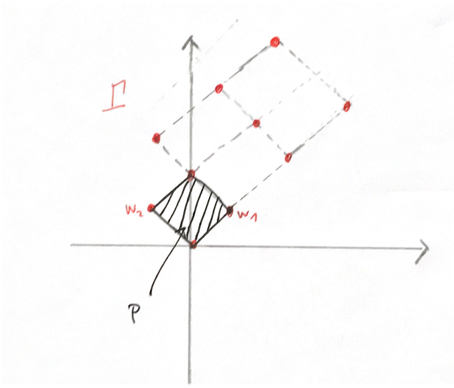

Abbildung 3: Sketch of Γ

$Q_z := z + P \subset \mathbb{C}$ and the homeomorphism $\pi_{|Q_z} : Q_z \to \pi(Q_z) \subset T$ with his inverse function $\phi_z : \pi(Q_z) \to Q_z$. We know claim that $\{\phi_z \mid z \in \mathbb{C}\}$ is acctually an complex atlas on T. The only thing remaining to prove is that the transiton maps are holomorphic. It follows for the transition map ψ_{zw} that for all $p \in Q_z$ $\pi(\psi_{zw}) = \pi(p) \Rightarrow p - \psi_{zw}(p) \in \Gamma$. And since Γ is discret we get that $p - \psi_{zw}$ is local constant $\Rightarrow \psi_{zw}$ is holomorphic. Analogously we get that ψ_{zw}^{-1} that is holomorphic.

Smooth affine plane curves

Let X be a closed subset of $\mathbb{C}^2$ defined as the locus of zeros of a polynomial $f \in \mathbb{C}[x, y]$; i.e. $X := \{(x, y) \in \mathbb{C}^2 \mid f(x, y) = 0\}$. Assume there is no

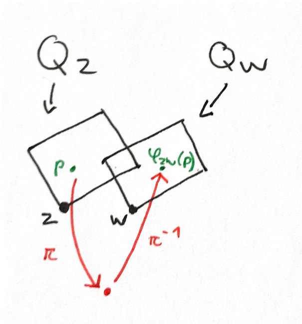

Abbildung 4: Qz and Q_w

point of X where the partial derivatives $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$ both vanish. We can then endow X with the structure of a Riemann surface as follows. In the neighbourhood of a point where $\frac{\partial f}{\partial y}$ is nonzero we define a complex chart by mapping a point to its $x-$coordinate; similary, for points where $\frac{\partial f}{\partial x}$ is nonzero we take the $y-$coordinate. By the inverse function theorem for holomorphic functions, in a small enough neighbourhood the above mappings are indeed homeomorphisms. Also the holomorphic version of the implicit function theorem implies that in in the points where x and y define a complex chart, the transition function from x to y is holomorphic, i.e. when $\frac{\partial f}{\partial y}$ does not vanish at some point, we may express y as a holomorphic function of x and where $\frac{\partial f}{\partial x}$ does not vanish at some point, we may express x as a holomorphic function of y. So we have defined a complex atlas.

Definition 3.1.4 (holomorphic map)

Let X and Y be Riemann surfaces. A holomorphic map $\psi : Y \to X$ is a continuous map such that for each pair of open subsets $U \subset X$, $V \subset Y$ satisfying $\psi(V) \subset U$ and complex charts $f : U \to \mathbb{C}$, $g : V \to \mathbb{C}$ the functions $f \circ \psi \circ g^{-1} : g(V) \to \mathbb{C}$ are holomorphic.

Part II.

In this part we always assume that the maps under consideration are nonconstant on all connected components, i.e. they do not map a whole component to a single point.

Proposition 3.2.1 (Local structure of holomorphic maps)

Let $\psi : Y \to X$ be a holomorphic map of Riemann surfaces, and y a point of Y with image $x = \psi(y)$ in X. There exist open neighbourhoods V_y of y satisfying $\psi(V_y) \subset U_x$, as well as complex charts $g_y : V_y \to \mathbb{C}$ and $f_x : U_x \to \mathbb{C}$ satisfying $f_x(x) = g_y(y) = 0$ such that the following diagramm commutes with an appropritate positive integer e_y that does not depend on the choice of the complex charts.

$$\begin{array}{ccc} V_y & \xrightarrow{g_y} & U_x \\ {\scriptstyle g_y}\downarrow & & \downarrow{\scriptstyle f_x} \\ \mathbb{C} & \xrightarrow{z \mapsto z^{e_y}} & \mathbb{C} \end{array}$$

Proof
Let $\psi : Y \to X$ be a holomorphic map of Riemann surfaces with $\psi(y) = x \in X$ for a point $y \in Y$. Definition 3.1.4 and may shrinking U_x and U_y gets us charts $g_y : V_y \to \mathbb{C}$ and $f_x : U_x \to \mathbb{C}$ with $x = \psi(y) \in U_x$ and $y \in V_y$ with $f_x(x) = g_y(y) = 0$. $f_x \circ \psi \circ g_y^{-1}$ is per Definition a holomorphic function, which vanishes at $0 \left((f_x \circ \psi \circ g_y^{-1})(0) = (f_x \circ \psi)(y) = f_x(x) = 0\right)$. As such it must be of the form $z \mapsto z^{e_y} \cdot H(z)$, where $H(z)$ is a holomorphic function with $H(0) \neq 0$ and $e_y \in \mathbb{N}$. Denote by *log* a fixed branch of the logarithm in a neighbourhood of $H(0)$. From complex analysis then we know (and by may again shrinking V_y) that the formula $h := \exp((\frac{1}{e_y}) \log(H))$ defines a holomorphic function h on $g_y(V_y)$ with: $h^{e_y} = \left(\exp((\frac{1}{e_y}) \cdot \log(H))\right)^{e_y} = \exp\left(\frac{\log(H)}{e_y} \cdot e_y\right) = \exp(\log(H)) = H$. Thus by replacing g_y by its composition with the map $\chi(z) = z \cdot h(z)$. We obtain the chart $\left(f_x \circ \psi \circ (\chi \circ g_x)^{-1}\right)(z) = z^{e_y}$ that satisfies the required properties.
The independence of e_y of the charts follows from the fact that changing a chart amounts to composing with an invertible holomorphic function.

Defintion 3.2.2 (ramification index)
The integer e_y of the Proposition 2.1.1 is called the ramification index or branching order of ψ at y. The points y with $e_y > 1$ are called branch points. In the follwoing we denote the set of branch points of ψ by S_ψ.

Corollary 3.2.3
A holomorphic map $f : X \to Y$ between Riemann surfaces is open (i.e. it maps open sets onto open sets)

Proof
Proposition 3.2.1 says that f looks like $z \mapsto z^k$ for an $k \in \mathbb{N}$ (at least lokaly). Since $z \mapsto z^k$ is holomorphic and as we assumed at the beginning of Part II. $z \mapsto z^k$ is non-constant, with the open mapping thereom we then get that f is indeed open.

Corollary 3.2.4
The fibres of ψ and the set S_ψ are discret subsets of Y

Proof
Otherwise there would exist $b \in Y$ such that the set $S := \{a \in X \mid \psi(a) = b\}$ has an accumualtion point. But then with the identity theorem we would get $\psi \equiv b$, which means ψ is constant, so therefore we have an contradiction.

Definition 3.2.5 (proper map)
Let X and Y be topological spaces. A continious map $f : X \to Y$ is called proper, if for every compact set $K \subset Y$ the preimage $f^{-1}(K) \subset X$ is compact.

Proposition 3.2.6
Let X be a connected Riemann surface and $\psi : Y \to X$ a proper holomorphic map. The map ψ is surjective with finite fibres and its restriction to $Y \setminus \psi^{-1}(\psi(S_\psi))$ is a finite topological cover of $X \setminus \psi(S_\psi)$

Proof
1) Finite fibres
Let $x \in X$ be a orbitrary point and consider the fibre $F_x := \psi^{-1}(x)$ and the compact set $K \subset Y$. Since ψ is proper the set $\psi^{-1}(K)$ with $F_x \subset \psi^{-1}(K)$ is compact. Corollary 3.2.4 then says $F_x \subset \psi^{-1}(K)$ is a discrete subset of a comapct set and therefore finite.

2) Surjective
The aim is to proof that $\psi(Y)$ is open an closed in X. Obviously Y is open in Y then by Corollary 3.2.3 $\psi(Y)$ is open in X. It remains to show that $\psi(Y) \subset X$ is closed, which is equivalent to prove $X \setminus \psi(Y)$ is open. Let $y \in X \setminus \psi(Y)$ be orbitary, then y has an open neighbourhood V with compact closure $\overline{V}$. As we know ψ is proper, so $\psi^{-1}(\overline{V})$ is compact. Let $E := Y \cap \psi^{-1}(\overline{V})$, then E is clearly compact and since ψ is continuous $\psi(E)$ is compact and therefore also closed. Let $U = V \setminus \psi(E)$. Then U is an open neighbourhood of y and it is disjoint from $\psi(Y)$. U is disjoint from $\psi(Y)$ cause if we suppose it is not then, there would exist a point $z \in U \cap \psi(Y)$ and a $c \in Y$ such that $\psi(c) = z$. This means $c \in \psi^{-1}(U) \subset \psi^{-1}(V) \subset \psi^{-1}(\overline{V})$. So $c \in Y \cap \psi^{-1}(\overline{V}) = E$. Therefore $z = \psi(c) \in \psi(E)$ which is a contradiction to $z \in U$. Thus $X \setminus \psi(Y)$ is open. All in all $\psi(Y)$ is closed and open. Since the only subsets of a connected topological space which are both closed and open are the empty set $\emptyset$ and the whole space. As X is connected it follows $\psi(Y) = X$. It remains to show that ψ is a covering map away from the branch points. Let for $y \in Y$ be $\psi(y) = x$. If x is not the image of a branch point, then each element of its fiber maps homeomorphically onto a neighbourhood of x since the bottom arrow of the commutative diagramm of Proposition 3.2.1 will be the identity map. As we now know the fibres are finite, its clear that ψ will be a covering map.
Lemma 3.2.8
Let X be a Riemann surface and $p : Y \to X$ a connected cover of X as a topological space. The space Y can be endowed with a unique complex structure for which p becomes a holomorphic mapping.
Proof
Existence:
Since $p : Y \to X$ is a connected cover of X each point $y \in Y$ has a neighbourhood V_i such that $p_{|V_i} : V_i \to U \subset X$ is an hoemorphism with $p(y) \in U$. If we take a complex chart $f : U' \to \mathbb{C}$ with $U' \subset U$, then the composition $(f \circ p)_i$ defines a complex chart in a neighbourhood of y. This way we obtain a complex atlas of Y by defining $\mathcal{V} := \{V_i \mid i \in I\}$ with the complex charts $(f \circ p)_i : V_i \to \mathbb{C}$.
Uniqueness:
Assume there is another complex atlas $\mathcal{V}'$ such that $p : (Y, \mathcal{V}') \to X$ is holomoprhic. Then the identity on $(Y, \mathcal{V})$ $id : (Y, \mathcal{V}) \to (Y, \mathcal{V}')$ is biholomorphic since locally $id(t) = (pr_{|V})^{-1} \circ pr(t)$ for a suitable open set V.
Proposition 3.2.9
Assume given a connected Riemann surface X, a discrete closed set S of points of X and a finite connected cover $\psi' : Y' \to X'$, where $X' := X \setminus S$. There exists a Riemann surface Y containing Y' as an open subset and a proper holomorphic map $\psi : Y \to X$ such that $\psi\,|_{Y'} = \psi'$ and $Y' = Y \setminus \psi^{-1}(S)$.
Proof
Fix a point $x \in S$ and also define the unitdisc $D := \{z \in \mathbb{C} \mid |z| < 1\} \subset \mathbb{C}$.
Since S is discret we find a connected open neighbourhood $x \in U_x$ of X such that $U_x \cap S = \emptyset$ and a complex chart $f : U_x \to D$ with $f(x) = 0$
Then the restriction $\psi'_{|\psi'^{-1}(U_x \setminus \{x\})}$ is a finite cover. Hence $\psi^{-1}(U_x \setminus \{x\}) := V_x^i$ decomposes as a finite disjoint union of connected components and each V_x^i is a cover of $U_x \setminus \{x\}$. Via the isomorphism $f_{|U_x \setminus \{x\}} U_x \setminus \{x\} \to \dot{D} := D \setminus \{0\}$ each V_x^i becomes by Example 2.4.12. (Talk 7) isomorphic to a cover $\dot{D} \to \dot{D}$ given

by $z \mapsto z^k$ for some $k > 1$. Now we choose points y_x^i for all i and x. We define Y as the disjoint union $Y := Y' \cup \{y_x^i\}$ and an extension ψ of ψ' to Y by:

$$\begin{aligned} \psi : Y := Y' \cup \{y_x^i\} &\to X \\ y \mapsto x \qquad & \text{if } y \in \{y_x^i\} \\ y \mapsto \psi'(y) \quad & \text{if } y \in Y' \end{aligned}$$

For each i and x we extend the holomorphic isomorphism $p_x^i : V_x^i \to \dot{D}$ to the bijection:

$$\begin{aligned} \overline{p_x^i} : V_x^i \cup \{y_x^i\} &\to D \\ y \mapsto p_y^i(y) \quad & \text{if } y \in V_x^i \\ y \mapsto 0 \text{ if } y \in \{y_x^i\} & \end{aligned}$$

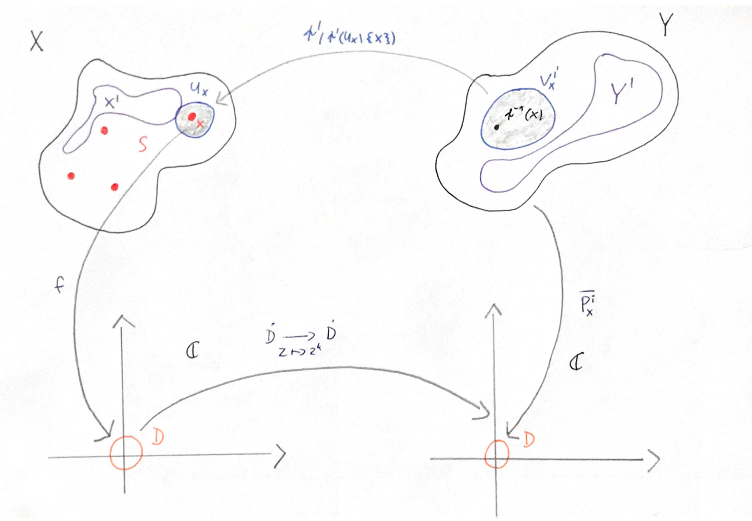

Abbildung 5: Illustration of the proof of Proposition 3.2.9

Together with the canonical complex structure on Y' defined in the proof of Lemma 3.2.8 $\{p_x^i\}$ form a complex atlas on Y. Also by Lemma 3.2.8 the map ψ is holomorphic. Finally the map ψ is proper, because by Example 3.2.5 (2) ψ' is proper and the fibres of ψ are finite, and the compact subsets of X' differ from those of X by finitely many points.

Theorem 3.2.7
In the situation of Proposition 3.2.6 mapping a Riemann surface $\psi : Y \to X$ over X to the topological cover $Y \setminus \psi^{-1}(S) \to X \setminus S$ obtained by restriction of ψ induces an equivalence of the category $Hol_{X,S}$ with the category of finite topological covers of $X \setminus S$.
Proof
In view of Proposition 3.2.9 it remains to prove that the functor of the theorem is fully faithful. This means the following: Let Y and Z be two Riemann surfaces equipped with proper holomorphic maps $\psi_Y : Y \to X$ and $\psi_Z : Z \to X$ with

all branch points above S and a morphism of covers $p' : Y' \to Z'$ over X' with $Y' = Y \setminus \psi_Y^{-1}(S)$ and $Z' = Z \setminus \psi_Z^{-1}(S)$, there is a unique holomorphic map $p : Y \to Z$ over X extending p'. As we know from Lemma 2.2.11 (Talk 5) the map $p' : Y' \to Z'$ is a cover, so it is holomorphic with respect to the unique complex structure on Y' by Lemma 3.2.8. Because of $\psi_Y |_{Y'} = \psi_Z \circ p'$ the complex structure of Y' must be compatible with the complex structure of Y. We now proceed similary as in the Proof of Proposition 3.2.9. Let $x \in S$ and $f : U_x \to D$ be a chart with $x \in U_x$ and $f(x) = 0$. Denote $U^* := U_x \setminus \{x\}$ and we assume that U_x is so small that ψ_Y and ψ_Z are unbranched over U^*. Then we define the connected components of $\psi_Y^{-1}(U_x)$ by $V_1, ..., V_n$ and the connected components of $\psi_Z^{-1}(U_x)$ by $W_1, ..., W_m$. Then $V_j^* := V_j \setminus \psi_Y^{-1}(x)$ are the connected components of $\psi_Y^{-1}(U^*)$ and $W_i^* := W_i \setminus \psi_Z^{-1}(x)$ are the connected compoents of $\psi_Z^{-1}(U^*)$ with $j \in \{1, ..., n\}$ and $i \in \{1, ..., m\}$. Since $p' \mid \psi_Y^{-1}(U^*) : \psi_Y^{-1}(U^*) \to \psi_Z^{-1}(U^*)$ is biholomorphic, we get that $n = m$ and by may renumbering we can set $p'(V_j^*) = W_j^*$. Since $\psi_Y \mid V_j^* : V_j^* \to U^*$ is a finite connected unbranched covering we get $V_j \cap \psi_Y^{-1}(x)$ (the same holds for $W_i \cap \psi_Z^{-1}(x)$) consists by Proposition 3.2.9 of exactly one point y_j (or z_i). Hence $p' \mid \psi_Y^{-1}(U^*) : \psi_Y^{-1}(U^*) \to \psi_Z^{-1}(U^*)$ can be continued to a bijective map $\psi_Y^{-1}(U) \to \psi_Z^{-1}(U)$ which assigns to the point y_j the point z_i. Since $\psi_Y : V_j \to U$ and $\psi_Z : W_i \to U$ are proper, the continuation is a homeomorphism and by appyling the Riemann's theorem on removable singularities (We can apply the Theorem cause as in the proof of Prop.3.2.9 mentioned V_j and W_i are isomorphic to $D := \{z \in \mathbb{C} \mid |z| < 1\}$) it is even biholomorphic. If we now apply this procedure for every single point on $s \in S$ we get the desired map $p : Y \to Z$.

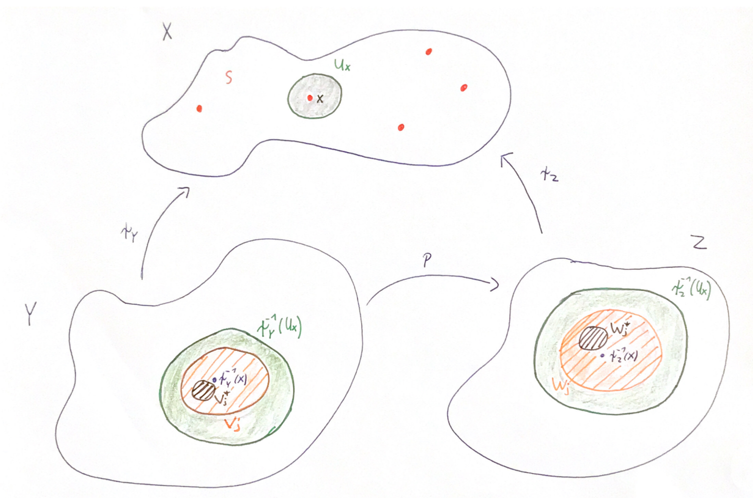

Abbildung 6: Constructing $p : Y \to Z$