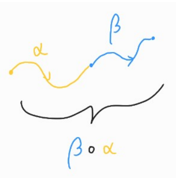
The monodromie action

Contents

Definition 1 (fundamental group). *The fundamental group for a topological space X and $x \in X$ is $\pi_1(X,x) = \{\text{loops in } X \text{ based at } x\}/\text{homotopy}$ with the multiplication being the composition*



and $c_x \colon [0,1] \to X t \mapsto x$ the neutral element

Definition 2 (simply connected). *A topological space is simply connected if it is path connected and if $\pi_1(X,x) \cong 1$ for all $x \in X$*

Construction 3 (monodromie action). *X topological space $x \in X$ $p \colon Y \mapsto X$ a cover.*
We define

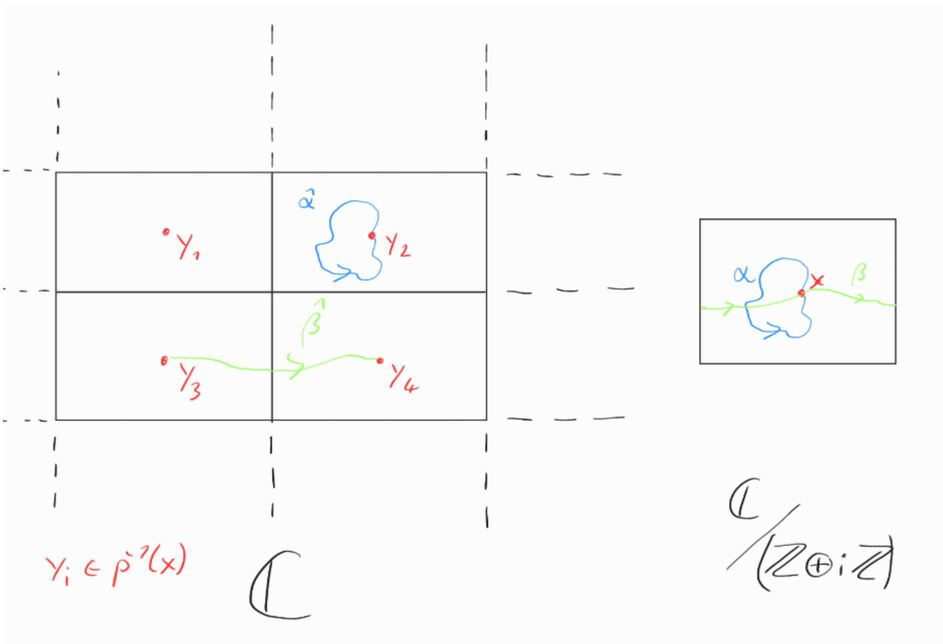
$$\pi_1(X,x) \times \pi_1(X,x) \to p^{-1}(x)$$
$$([\alpha], y) \mapsto [\alpha]y := \hat{\alpha}(1)$$

with $\hat{\alpha}$ a lift of α along p with $\hat{\alpha}(0) = y$
That $\hat{\alpha}$ is a lift means that $p \circ \hat{\alpha} = \alpha$ or that the following diagram is commutative:

$$\begin{array}{ccc} & & Y \\ & \overset{\hat{\alpha}}{\nearrow} & \downarrow p \\ [0,1] & \xrightarrow{\alpha} & X \end{array}$$

It is not clear that this construction is well defined. But we will prove that in the next Lemma.

Example 4. *We are looking at the canonical map $\pi \colon \mathbb{C} \to \mathbb{C}/(\mathbb{Z} \oplus i\mathbb{Z})$ this is a cover so we can look at the monodromie action. We chose some x in $\mathbb{C}/(\mathbb{Z} \oplus i\mathbb{Z})$ and a loop α. We can see that we only get a lift $\hat{\alpha}$ that is not a loop if α is not contractible. This will follow as easy Corollary of the next Lemma.*



Lemma 5 (Lemma 2.3.2 in the book)**.** *Let $p\colon Y \to X$ be a cover $x \in X$ and $y \in p^{-1}(x)$*

1. *Given a path $\alpha\colon [0,1] \to X$ with $\alpha(0) = x$ there is a unique path $\hat{\alpha}\colon [0,1] \to Y$ with $\hat{\alpha}(0) = y$ and $p \circ \hat{\alpha} = \alpha$*

2. *Assume moreover that we are given a second path $\beta\colon [0,1] \to X$ homotopic to α. Then the unique lift $\hat{\beta}\colon [0,1] \to X$ with $\hat{\beta}(0) = y$ and $p \circ \hat{\beta} = pb$ has the same endpoint as $\hat{\alpha}$.*

Proof. We want to construct a path $\hat{\alpha}$ as in the following diagram

$$\begin{array}{ccc} & & Y \\ & \overset{\hat{\alpha}}{\nearrow} & \downarrow p \\ [0,1] & \xrightarrow{\alpha} & X \end{array}$$

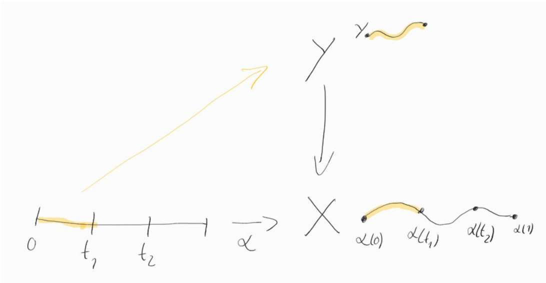
For statement 1) uniqueness follows from Proposition 2.2.2

existence: It is easy to see that such a lift exists if p is a trivial cover

$$\begin{array}{ccc} & & \coprod X \\ & \overset{\hat{\alpha}}{\nearrow} & \downarrow p \\ [0,1] & \xrightarrow{\alpha} & X \end{array}$$

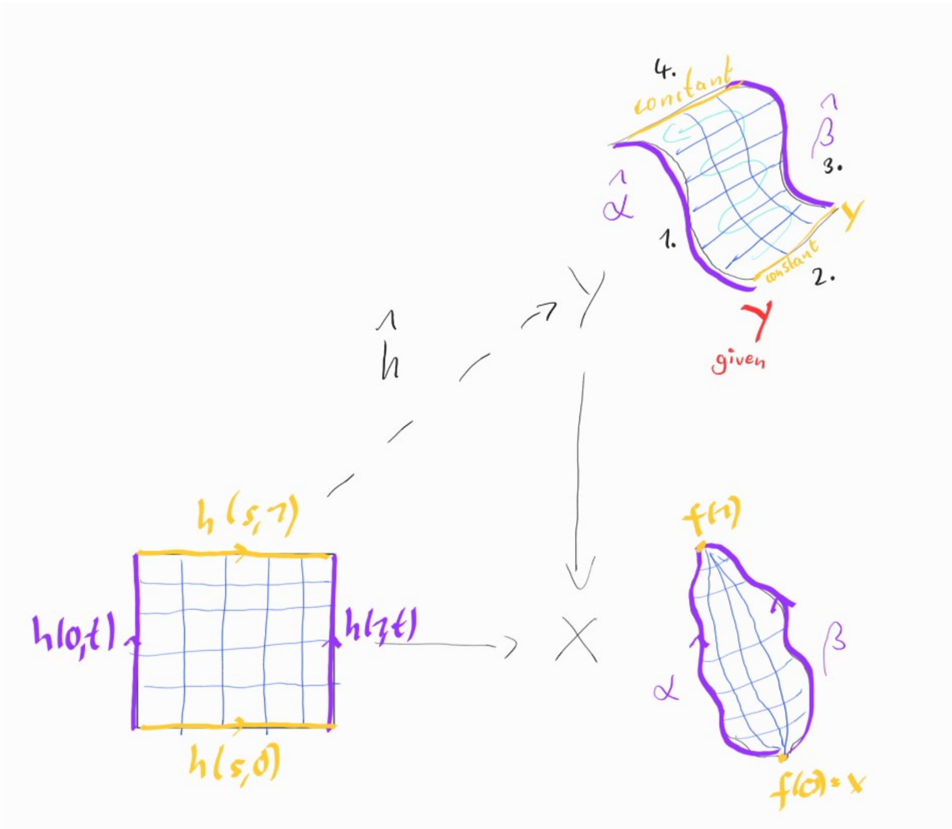
Because every cover is locally isomorphic to a trivial cover we can use this fact to prove the theorem for all covers.
For every $x \in \alpha([0,1])$ we choose a set V_x so that $p_{|p^{-1}(V_x)}\colon p^{-1}(V_x) \to V_x$ is isomorphic to a trivial cover.
The sets $\alpha^{-1}(V_x)$ form a covering of the interval $[0,1]$
Because the interval $[0,1]$ is compact we get a finite subcovering.
Hence we can choose a subdivision $0 = t_0 \leq t_1 \leq \cdots \leq t_n = 1$ of $[0,1]$

such that each closed interval $[t_i, t_{i+1}]$ is contained in some $\alpha^{-1}(V_x)$. Thus the cover is trivial over each $\alpha([t_i, t_{i+1}])$. We can now construct $\tilde{\alpha}$ inductively.



For statement 2) We will show that we can lift homotopies that means for a homotopy $h\colon [0,1] \times [0,1] \to X$ with $h(0,t) = \alpha(t)$ and $h(1,t) = \beta(t)$ there is a homotopy $\hat{(h)} : [0,1] \times [0,1] \to Y$ with $p \circ \hat{(h)} = h$, $\hat{h}(0,t) = \hat{\alpha}(t)$ and $\hat{h}(1,t) = \hat{\beta}(t)$. The construction of $\hat{(h)}$ is similar to that of $\hat{\alpha}$ in the way that we again divide the square $[0,1] \times [0,1]$ in smaller parts that we then lift individually. Similar to above we choose subdivisions $0 = t_0 \leq t_1 \leq \cdots \leq t_n = 1$ and $0 = s_0 \leq s_1 \leq \cdots \leq s_m = 1$ of $[0,1]$ so we can lift h restricted to $[t_i, t_{i+1}] \times [s_j, s_{j+1}]0$. That this functions is assured by the Lebesgue lemma. Then we lift these smaller squares one after the other going serpentwise from the border $h(0,1)$, where we know that $\hat{h}$ should fulfil $\hat{h}(0,0) = \alpha(0) = y$, to the top of the big square.

The lifts of the small squares fit together at the borders because they coincide at one corner and such the lifing on then intersection is unique. So we get a $\hat{h}$ as in the following picture.



1. We know now that $\hat{\alpha}$ is the left border of $\hat{h}$ because as we showed in the first statement lifts with a fixed point in Y are unique and $\hat{\alpha}$ is one possibility for a lift.
2. analogue we get that the bottom border should be the constant path
3. with that we get that the right border is $\hat{\beta}$
4. finally we get that the top border is constant

Therefore it follows that $\hat{\beta}(1) = \hat{\alpha}(1)$. □

Definition 6. *X a topological space $x \in X$*
We define a functor.

$$Fib_x\colon \{Covers\, of\, X\} \to \{left\, \pi_1(X, x)\, sets\}$$
$$(p\colon Y \to X) \mapsto p^{-1}(x)$$

Which maps funtions between covers of X like this:
$p\colon Z \to X, q\colon Z \to X$ *covers over X*

$f\colon Z \to Y$ *a map of covers over* X *then* $Fib_x(f)$ *is the map* $f_{|p^{-1}(x)} p^{-1}(x) \to q^{-1}(x)$
This map is well defined between sets because f is a map over X and a map of left $\pi_1(X,x)$ *sets because for* $[\alpha] \in \pi_1(X,x)$ *and* $y \in p^{-1}(x)$ $f([\alpha]y) = [f \circ \alpha] f(y)$

Definition 7. *A topological space X is locally simply connected if every point has a basis of simply connected open neighborhoods.*

Theorem 8 (2.3.4 in the book)**.** *Let X be a connected and locally simply connected topological space and* $x \in X$ *a base point.*
The functor Fib_x *induces an equivalence of the category of covers of X with the category of left* $\pi_1(X,x)$ *sets. Connected covers correspond to* $\pi_1(X,x)$ *sets with transitive action and Galois covers to coset spaces of normal subgroups.*

The next two theorems we will only state because they are proven in the next chapter of the book and thus covered in the next talk.

Theorem 9 (2.3.5 in the book)**.** *For a connected and locally simply connected topological space X and a base point* $x \in X$ *the functor* Fib_x *is representable by a cover* $\tilde{X}_x \to X$. *That means* $Fib_x \cong Hom_X(\tilde{X}_x, -)$

Remark. $Aut(\tilde{X}_x \mid X)$ *gives a right action on* $Fib_x(Y) \cong Hom_X(\tilde{X}_x, Y)$ *for Y a cover of X.*

$$Hom_X(\tilde{X}_x, Y) \times Aut(\tilde{X}_x, X) \to Hom_X(\tilde{X}_x, Y) \quad (1)$$
$$(f, \phi) \mapsto f \circ \phi \quad (2)$$

Then we get a left action by going to $Aut(\tilde{X}_x \mid X)^{op}$ *where the multiplication is swapped.*

Theorem 10 (2.3.7 in the book)**.** *The cover* $\tilde{X}_x$ *is a connected Galois cover of X, with automorphism group isomorphic to* $\pi_1(X,x)$. *Moreover for each cover* $Y \to X$ *the left action of* $Aut(\tilde{X}_x \mid X)^{op}$ *on* $Fib_x(X)$ *given by the previous remark is exactly the monodromy action of* $\pi_1(X,x)$.

Proof of Theorem 8. First we want to show that the following functor is an equivalence of categories.

$$Fib_x\colon \{Covers\, of\, X\} \to \{left\, \pi_1(X,x)\, sets\}$$
$$(p\colon Y \to X) \mapsto p^{-1}(x)$$

For that we will prove that Fib_x is fully faithful and essentially surjective.
$\underline{fully\, faithfulness:}$

Let $p\colon Y \to X$ and $q\colon Z \to X$ be two covers of X. We will show that every map $\phi\colon Fib_x(Y) \to Fib_x(Z)$ comes from a unique map $Y \to Z$ over X. We assume Y, Z are connected. Otherwise we will show that for every connected component of Z and Y and then fit the map together. We have an isomorphism

$$Fib_x(Y) \cong Hom_X(\tilde{X}_x, Y)$$
$$y \mapsto \pi_y$$

by theorem 9 Now we choose an element $y_0 \in Fib_x(Y)$ By theorem 2.2.10 in the book we get an isomorphism $\tilde{\pi_{y_0}}\colon U_{y_0}\backslash\tilde{X}_x \to Y$ with $U_{y_0} = Aut(\tilde{X}_x \mid Y)$ because $\tilde{X}_x \to X$ is Galois by 10
With the isomorphism:

$$Fib_x(Z) \cong Hom_X(\tilde{X}_x, Z)$$
$$z \mapsto \pi_z$$

we get a commutative diagram:

$$\begin{array}{ccc} Fib_x(y) & = & Hom_X(\tilde{X}_x, Y) \\ \downarrow{\scriptstyle\phi} & & \downarrow \\ Fib_x(Z) & = & Hom_X(\tilde{X}_x, Z) \end{array}$$

and a element $\pi_{z_0} \in Hom_X(\tilde{X}_x)$ induced by the above defined element $y_0 \in Fib_x(Y)$. The map $\pi_{z_0}\colon \tilde{X}_x \to Z$ induces with the universal property of the quotient a map $\tilde{\pi}_{z_0}\colon U_{y_0}\backslash\tilde{X}_x \to Z$ because U_{y_0} injects into the stabilizer of $\phi(y_0)$ via ϕ. By composition we get a map $\tilde{\pi}_{z_0} \circ \tilde{\pi}_{y_0}^{-1}\colon Y \to U_{y_0}\backslash\tilde{X}_x \to Z$ as we wanted. Such a map is unique because if we had two maps $f, g\colon Y \to Z$ between connected covers that coincide restricted to the fibre of x they are the same by Proposition 2.2.2 of the book.

essential surjective For every $\pi_1(X,x)$left set S we have to find a cover that exhibits S as fibre of x. For S transitive we may take the quotient $\pi_1(X,x)_S\backslash\tilde{X}_x$ that makes sense because by Theorem 10 $\pi_1(X,x)_S \sup Aut_X(\tilde{X}_x)$ then we define the cover $q\colon \pi_1(X,x)_S\backslash\tilde{X}_x \to X$. Furthermore $q^{-1}(x) = S$. That is the case because for $p\colon \tilde{X}_x \to X$ the universal cover $p{-}1(x) \cong Aut(\tilde{X}_x)$ Hence $q^{-1} \cong Aut(\tilde{X}_x)/\pi_1(X,x)_S \cong \pi_1(X,x)/\pi_1(X,x)_S \cong \pi_1(X,x){\cdot}\, S = S$ because S is transitive. □

Corollary 11. *Let X be a connected and locally simply connected topological space and $x \in X$. Then the functor Fib_x induces an equivalence of the category of finite covers of X with the category of finite continous left $\pi_1(\hat{X},x)$-sets. Connected covers correspond to finite left $\pi_1(\hat{X},x)$-sets with transitive action and Galois covers to coset spaces of open normal subgroups. Here $\pi_1(\hat{X},x)$ denotes the profinite completion of $\pi_1(X,x)$as in Example 1.3.4(2) in the book*