Rigid Categories and Berkovich Motives

Giovanni Rossanigo

July 7, 2025

Abstract

These are the notes for my talk in the *HIOB* seminar of summer semester 2025. Our first goal is to study the dualizable objects for the Lurie tensor product on $\mathrm{Pr}^{\mathrm{L}}_{\mathrm{st}}$, the $(\infty,1)$-category of stable presentable $(\infty,1)$-categories and left adjoints. We show that the dualizable objects coincide with the *compactly assembled* $(\infty,1)$*-categories*. These are a suitable generalization of compactly generated $(\infty,1)$-categories, where the notion of "being generated under filtered colimits by compact objects" is replaced by "being generated under filtered colimits by compact exhaustible objects". We then introduce *rigid* $(\infty,1)$*-categories*. We view them as presentably symmetric monoidal $(\infty,1)$-categories that are as close as they could be to being "rigidly-compactly generated" without being compactly generated. We finally discuss Scholze's results: we show that for a sufficiently nice arc-sheaf X the $(\infty,1)$-category $\mathcal{D}_{\mathrm{mot}}(X)$ of motivic sheaves over X is rigid over Sp. We deduce then a categorical version of the Künneth formula.

1 Some remarks

Let's start with a long list of observations.

Remark 1.1. Recall that an $(\infty,1)$-category $\mathcal{C}$ is *presentable* if it is accessible (that is, there exists some small regular cardinal κ such that $\mathcal{C} \simeq \mathrm{Ind}_\kappa(\mathcal{D})$ is the ind-completion[1] of a small $(\infty,1)$-category $\mathcal{D}$) and admits small colimits. Let Pr^{L} denote the $(\infty,1)$-category of presentable $(\infty,1)$-categories and colimit-preserving functors (or equivalently, left adjoints). Lurie constructed in [Lur17, Section 4.8] a symmetric monoidal structure on Pr^{L} characterized by the following universal property: given two presentable $(\infty,1)$-categories $\mathcal{C}$ and $\mathcal{D}$, their Lurie tensor product is a presentable $(\infty,1)$-category $\mathcal{C}\otimes\mathcal{D}$ with a functor $\mathcal{C}\times\mathcal{D}\to\mathcal{C}\otimes\mathcal{D}$ such that for every presentable $(\infty,1)$-category $\mathcal{E}$, precomposition with it induces an equivalence

$$\mathrm{Fun}^{\mathrm{L}}(\mathcal{C}\otimes\mathcal{D},\mathcal{E}) \to \mathrm{Fun}^{\mathrm{L,L}}(\mathcal{C}\times\mathcal{D},\mathcal{E}).$$

Here the superscript L, L denotes the full subcategory spanned by those functors $\mathcal{C}\times\mathcal{D}\to\mathcal{E}$ which preserve colimits separately in each variable. Moreover, he showed that we can compute the tensor product via

$$\mathcal{C}\otimes\mathcal{D} \simeq \mathrm{RFun}(\mathcal{C}^{\mathrm{op}},\mathcal{D})$$

the $(\infty,1)$-category of right adjoints $\mathcal{C}^{\mathrm{op}}\to\mathcal{D}$. Moreover, this symmetric monoidal structure is also closed:

[1] In general, we define $\mathrm{Ind}_\kappa(\mathcal{D})$ as the full subcategory of $\mathrm{Fun}(\mathcal{D}^{\mathrm{op}},\mathrm{Spc})$ generated by representables under κ-filtered colimits. Recall also that, if $\mathcal{D}$ admits κ-small colimits, then $\mathrm{Ind}_\kappa(\mathcal{D}) \simeq \mathrm{Fun}^{\kappa\text{-limit}}(\mathcal{D}^{\mathrm{op}},\mathrm{Spc})$ consists precisely of those functors $\mathcal{D}^{\mathrm{op}}\to\mathrm{Spc}$ which preserve κ-small limits.

for every pair of presentable $(\infty,1)$-categories $\mathcal{C}$ and $\mathcal{D}$, we have a natural equivalence

$$\mathrm{Fun}^{\mathrm{L}}(\mathcal{C}\otimes\mathcal{D},\mathcal{E}) \simeq \mathrm{Fun}^{\mathrm{L},\mathrm{L}}(\mathcal{C}\times\mathcal{D},\mathcal{E}) \simeq \mathrm{Fun}^{\mathrm{L}}(\mathcal{C},\mathrm{Fun}^{\mathrm{L}}(\mathcal{D},\mathcal{E}))$$

in $\mathcal{E}\in\mathrm{Pr}^{\mathrm{L}}$, so that $\mathrm{Fun}^{\mathrm{L}}(-,-)$ exhibits the internal hom. Since $\mathrm{Fun}^{\mathrm{L}}(\mathrm{Spc},\mathcal{C})\simeq\mathcal{C}$, we see that the $(\infty,1)$-category of animae Spc is the neutral element for the Lurie tensor product.

Remark 1.2. We can apply the business of symmetric monoidal $(\infty,1)$-category theory to $(\mathrm{Pr}^{\mathrm{L}},\otimes,\mathrm{Spc})$. We can construct the $(\infty,1)$-category of *commutative algebra objects*[2] $\mathrm{CAlg}(\mathrm{Pr}^{\mathrm{L}})$ and for every commutative algebra object $\mathcal{B}\in\mathrm{CAlg}(\mathrm{Pr}^{\mathrm{L}})$ an $(\infty,1)$-category of $\mathcal{B}$-*modules* $\mathrm{Mod}_{\mathcal{B}}(\mathrm{Pr}^{\mathrm{L}})$ which inherits a closed symmetric monoidal structure from $(\mathrm{Pr}^{\mathrm{L}},\otimes,\mathrm{Spc})$. More explicitly, we can think of a commutative algebra object as a triple $(\mathcal{B},\mu,\eta)$ where $\mu:\mathcal{B}\otimes\mathcal{B}\to\mathcal{B}$ and $\eta:\mathrm{Spc}\to\mathcal{B}$ are colimit-preserving functors such that the following diagrams

$$\begin{array}{ccc} \mathrm{Spc}\otimes\mathcal{B} & \xrightarrow{\eta\otimes\mathrm{id}_{\mathcal{B}}} & \mathcal{B}\otimes\mathcal{B} \\ \downarrow & \swarrow_{\mu} & \\ \mathcal{B} & & \end{array} \qquad \begin{array}{ccc} \mathcal{B}\otimes\mathcal{B}\otimes\mathcal{B} & \xrightarrow{\mu\otimes\mathrm{id}_{\mathcal{B}}} & \mathcal{B}\otimes\mathcal{B} \\ {\scriptstyle \mathrm{id}_{\mathcal{B}}\otimes\mu}\downarrow & & \downarrow{\scriptstyle\mu} \\ \mathcal{B}\otimes\mathcal{B} & \xrightarrow[\mu]{} & \mathcal{B} \end{array}$$

commute (up to higher coherence conditions). They are called the multiplication and unit map,respectively. Similarly, a $\mathcal{B}$-module is given by a presentable $(\infty,1)$-category $\mathcal{M}$ with a colimit-preserving functor $a:\mathcal{B}\otimes\mathcal{M}\to\mathcal{M}$, called the action, such that the following diagrams

$$\begin{array}{ccc} \mathrm{Spc}\otimes\mathcal{M} & \xrightarrow{\eta\otimes\mathrm{id}_{\mathcal{M}}} & \mathcal{B}\otimes\mathcal{M} \\ \downarrow & \swarrow_{a} & \\ \mathcal{M} & & \end{array} \qquad \begin{array}{ccc} \mathcal{B}\otimes\mathcal{B}\otimes\mathcal{M} & \xrightarrow{\mu\otimes\mathrm{id}_{\mathcal{M}}} & \mathcal{B}\otimes\mathcal{M} \\ {\scriptstyle \mathrm{id}_{\mathcal{M}}\otimes\mu}\downarrow & & \downarrow{\scriptstyle a} \\ \mathcal{B}\otimes\mathcal{M} & \xrightarrow[a]{} & \mathcal{M} \end{array}$$

commute (up to higher coherence conditions).

Remark 1.3. There is also a stable version of this construction. Let Sp denote the $(\infty,1)$-category of spectra, defined as the colimit

$$\mathrm{Sp}=\mathrm{colim}(\mathrm{Spc}_* \xrightarrow{\Sigma} \mathrm{Spc}_* \xrightarrow{\Sigma} \dots)$$

in Pr^{L}, or dually, as the limit

$$\mathrm{Sp}\simeq\lim(\dots \xrightarrow{\Omega} \mathrm{Spc}_* \xrightarrow{\Omega} \mathrm{Spc}_*)$$

in Pr^{R}, the $(\infty,1)$-category of presentable $(\infty,1)$-categories and right adjoints. Here Spc_* denotes the $(\infty,1)$-category of pointed spaces, whereas Σ and Ω denote the suspension and loop functor, respectively. By adjoining a point to an anima and then "infinitely suspending it" we obtain a functor $\Sigma^\infty_+:\mathrm{Spc}\to\mathrm{Sp}$. Thanks to this functor, one can easily show that a presentable $(\infty,1)$-category $\mathcal{C}$ is stable if and only if the canonical map

$$\mathrm{id}_{\mathcal{C}}\otimes\Sigma^\infty_+:\mathcal{C}\to\mathcal{C}\otimes\mathrm{Sp}$$

is an equivalence. In particular, since Sp is a stable $(\infty,1)$-category, we can use the inverse of the equivalence $\mathrm{Sp}\to\mathrm{Sp}\otimes\mathrm{Sp}$ to make Sp into a commutative algebra of Pr^{L}. We conclude that the $(\infty,1)$-category of stable presentable $(\infty,1)$-categories and left adjoints can be realized as

$$\mathrm{Pr}^{\mathrm{L}}_{\mathrm{st}}\simeq\mathrm{Mod}_{\mathrm{Sp}}(\mathrm{Pr}^{\mathrm{L}})$$

[2] The objects of $\mathrm{CAlg}(\mathrm{Pr}^{\mathrm{L}})$ are also called *presentably symmetric monoidal* $(\infty,1)$*-categories*.

and that taking spectrum objects $-\otimes \mathrm{Sp} : \mathrm{Pr}^{\mathrm{L}} \to \mathrm{Pr}^{\mathrm{L}}_{\mathrm{st}}$ defines a symmetric monoidal functor.

Remark 1.4. One feature of having a symmetric monoidal structure on a symmetric monoidal $(\infty,1)$-category $(\mathcal{C}, \otimes, \mathbb{1})$ is that we can look at those objects which are finite in the $\otimes$-structure. Recall that an object $x \in \mathcal{C}$ is called *dualizable* if there exists an object $x^\vee \in \mathcal{C}$ such that $x^\vee \otimes_{\mathcal{C}} - \dashv x \otimes_{\mathcal{C}} -$ are adjoint functors $\mathcal{C} \to \mathcal{C}$. Equivalently, x is dualizable if there exists an object $x^\vee$ and maps $c : \mathbb{1}_{\mathcal{C}} \to x^\vee \otimes_{\mathcal{C}} x$ and $e : x \otimes_{\mathcal{C}} x^\vee \to \mathbb{1}_{\mathcal{C}}$, called the *coevaluation* and *evaluation*, such that the composites

$$x \simeq x \otimes_{\mathcal{C}} \mathbb{1}_{\mathcal{C}} \xrightarrow{\mathrm{id}_x \otimes c} x \otimes_{\mathcal{C}} x^\vee \otimes_{\mathcal{C}} x \xrightarrow{e \otimes \mathrm{id}_x} x, \qquad x^\vee \simeq \mathbb{1}_{\mathcal{C}} \otimes_{\mathcal{C}} x^\vee \xrightarrow{c \otimes \mathrm{id}_{x^\vee}} x^\vee \otimes_{\mathcal{C}} x \otimes_{\mathcal{C}} x^\vee \xrightarrow{\mathrm{id}_{x^\vee} \otimes e} x^\vee$$

are equivalent to the respective identities.

We can look at the dualizable objects in $(\mathrm{Pr}^{\mathrm{L}}_{\mathrm{st}}, \otimes, \mathrm{Sp})$. Recall that a presentable $(\infty,1)$-category $\mathcal{C}$ is called *compactly generated* if there exists a small $(\infty,1)$-category $\mathcal{C}_c$ with finite colimits and an equivalence $\mathcal{C} \simeq \mathrm{Ind}(\mathcal{C}_c)$. In this case $\mathcal{C}$ is stable if and only if its *category of compact objects* $\mathcal{C}_c$ is stable.

Remark 1.5. Let $\mathcal{C} \in \mathrm{Pr}^{\mathrm{L}}_{\mathrm{st}}$ be a stable compactly generated $(\infty,1)$-category. We claim that $\mathcal{C}$ is dualizable for the Lurie tensor product and that its dual $\mathcal{C}^\vee$ can be identified with $\mathrm{Ind}(\mathcal{C}_c^{\mathrm{op}})$. For this, it is useful to remember that small stable $(\infty,1)$-categories are enriched over Sp: one can define a mapping spectrum

$$\mathrm{map}_{\mathcal{C}}(-,-) : \mathcal{C}^{\mathrm{op}} \times \mathcal{C} \to \mathrm{Sp}$$

such that taking the underlying space $\Omega^\infty \mathrm{map}_{\mathcal{C}}(-,-) \simeq \mathrm{Hom}_{\mathcal{C}}(-,-)$ gives back the mapping anima of $\mathcal{C}$. The duality data are then easily constructed. We define the evaluation morphism

$$e : \mathrm{Ind}(\mathcal{C}_c^{\mathrm{op}}) \otimes \mathrm{Ind}(\mathcal{C}_c) \simeq \mathrm{Ind}(\mathcal{C}_c^{\mathrm{op}} \otimes^{\mathrm{Rex}} \mathcal{C}_c) \to \mathrm{Sp}$$

by ind-extending the mapping spectrum (and using that $\mathrm{Ind}(\mathcal{C}^{\mathrm{op}}) \otimes \mathrm{Ind}(\mathcal{C}_c)$ can be identified with the ind-completion of $\mathcal{C}_c^{\mathrm{op}} \otimes^{\mathrm{Rex}} \mathcal{C}_c$ for a suitable tensor product of small idempotent-complete stable $(\infty,1)$-categories). We also define the coevaluation morphism

$$c : \mathrm{Sp} \to \mathrm{Ind}(\mathcal{C}^{\mathrm{op}}) \otimes \mathrm{Ind}(\mathcal{C}_c) \simeq \mathrm{Ind}(\mathcal{C}_c^{\mathrm{op}} \otimes^{\mathrm{Rex}} \mathcal{C}_c)$$

by just picking the mapping spectrum $\mathrm{map}_{\mathcal{C}}(-,-)$. Checking the triangle identities is then a matter of unwinding the definitions.

It turns out that not every dualizable object in $(\mathrm{Pr}^{\mathrm{L}}_{\mathrm{st}}, \otimes, \mathrm{Sp})$ is compactly generated.

Definition 1.6. Let $\mathcal{C}$ be an accessible $(\infty,1)$-category with filtered colimits. We will say that $\mathcal{C}$ is *compactly assembled* if the colimit functor $\mathrm{colim} : \mathrm{Ind}(\mathcal{C}) \to \mathcal{C}$ has a left adjoint.

Because then we can prove the following result due to Lurie [Lur18, Proposition D.7.3.1] (even though the formulation we are presenting is taken from [AK24, Theorem 2.2.15 and Theorem 2.9.2]).

Theorem 1.7. Let $\mathcal{C}$ be a presentable $(\infty,1)$-category. Unstably, we have that the following are equivalent assertions.

(1) The $(\infty,1)$-category $\mathcal{C}$ is compactly assembled.

(2) The $(\infty,1)$-category $\mathcal{C}$ is a retract in Pr^{L} of a compactly generated category.

Stably, that is if $\mathcal{C}$ is also stable, the previous assertions are equivalent to the following.

(3) The $(\infty, 1)$-category $\mathcal{C}$ is dualizable as an object of $\mathrm{Pr}^{\mathrm{L}}_{\mathrm{st}}$.

Proof. We don't discuss the implication $(1) \Rightarrow (2)$ and we refer the reader to Lurie's proof (or to [AK24, Section 2.2, 2.3]). We prove $(2) \Rightarrow (1)$. Assume first that $\mathcal{C}$ is compactly generated, so that we can write $\mathcal{C} \simeq \mathrm{Ind}(\mathcal{C}_c)$. Then the universal property of ind-extensions provides us a functor

$$\begin{array}{ccc} \mathcal{C}_c & \xrightarrow{\;y_{\mathcal{C}}\;} & \mathrm{Ind}(\mathcal{C}) \\ {\scriptstyle y_{\mathcal{C}}}\big\downarrow & & \big\downarrow{\scriptstyle y_{\mathrm{Ind}(\mathcal{C})}} \\ \mathrm{Ind}(\mathcal{C}) & \underset{\hat{y}_{\mathcal{C}}}{\dashrightarrow} & \mathrm{Ind}(\mathrm{Ind}(\mathcal{C})) \end{array}$$

which preserves colimits. One verifies easily that this is the required left adjoint to $\mathrm{colim}_{\mathcal{C}}$. For the general case, let $\mathcal{D}$ be compactly generated and assume that we are given a retract $\mathcal{C} \xrightarrow{i} \mathcal{D} \xrightarrow{r} \mathcal{C}$ in Pr^{L}. We can then consider the diagram

$$\begin{array}{ccccc} \mathrm{Ind}(\mathcal{C}) & \xrightarrow{\mathrm{Ind}(i)} & \mathrm{Ind}(\mathcal{D}) & \xrightarrow{\mathrm{Ind}(r)} & \mathrm{Ind}(\mathcal{C}) \\ {\scriptstyle \mathrm{colim}_{\mathcal{C}}}\big\downarrow & & {\scriptstyle \hat{y}_{\mathcal{D}}}\big\uparrow\big\downarrow{\scriptstyle \mathrm{colim}_{\mathcal{D}}} & & \big\downarrow{\scriptstyle \mathrm{colim}_{\mathcal{C}}} \\ \mathcal{C} & \xrightarrow[i]{} & \mathcal{D} & \xrightarrow[r]{} & \mathcal{C} \end{array}$$

where the horizontal raws compose to the identity and whose squares commute. We wish to construct a left adjoint $\hat{y}_{\mathcal{C}}$ to $\mathrm{colim}_{\mathcal{C}}$. Take $x \in \mathcal{C}$ and $y \in \mathrm{Ind}(\mathcal{C})$. We have then morphisms

$$\begin{aligned} \mathrm{Hom}_{\mathcal{C}}(x, \mathrm{colim}_{\mathcal{C}}\, y) &\xrightarrow{i} \mathrm{Hom}_{\mathcal{D}}(i(x), i(\mathrm{colim}_{\mathcal{C}}\, y)) \\ &\simeq \mathrm{Hom}_{\mathcal{D}}(i(x), \mathrm{colim}_{\mathcal{D}}(\mathrm{Ind}(i)(y))) \\ &\simeq \mathrm{Hom}_{\mathrm{Ind}(\mathcal{D})}(\hat{y}_{\mathcal{D}}(i(x)), \mathrm{Ind}(i)(y)) \\ &\xrightarrow{\mathrm{Ind}(r)} \mathrm{Hom}_{\mathrm{Ind}(\mathcal{C})}(\mathrm{Ind}(r)(\hat{y}_{\mathcal{D}}(i(x))), \mathrm{Ind}(r)(\mathrm{Ind}(i)(y))) \\ &\simeq \mathrm{Hom}_{\mathrm{Ind}(\mathcal{C})}(\mathrm{Ind}(r)(\hat{y}_{\mathcal{D}}(i(x))), y) \\ &\xrightarrow{\mathrm{colim}_{\mathcal{C}}} \mathrm{Hom}_{\mathrm{Ind}(\mathcal{C})}(\mathrm{colim}_{\mathcal{C}}(\mathrm{Ind}(r)(\hat{y}_{\mathcal{D}}(i(x)))), \mathrm{colim}_{\mathcal{C}}\, y) \\ &\simeq \mathrm{Hom}_{\mathcal{C}}(x, \mathrm{colim}_{\mathcal{C}}\, y) \end{aligned}$$

which compose to the identity. Here the first map is induced by functoriality of i, the second one by commutativity of the left square in the previous diagram, the third one by adjunction $\hat{y}_{\mathcal{D}} \dashv \mathrm{colim}_{\mathcal{D}}$, the fourth one by functoriality of $\mathrm{Ind}(r)$, the fifth one by using that , $\mathrm{Ind}(r) \circ \mathrm{Ind}(i) \simeq \mathrm{id}_{\mathrm{Ind}(\mathcal{C})}$, the sixth one by functoriality of $\mathrm{colim}_{\mathcal{C}}$ and the last one by using that $\hat{y}_{\mathcal{D}}$ is fully-faithful (being the Yoneda fully-faithful), that is

$$\mathrm{colim}_{\mathcal{C}} \circ \mathrm{Ind}(r) \circ \hat{y}_{\mathcal{D}} \circ i \simeq r \circ \mathrm{colim}_{\mathcal{D}} \circ \hat{y}_{\mathcal{D}} \circ i \simeq r \circ i \simeq \mathrm{id}_{\mathcal{C}}.$$

Anyway, we have exhibited $\mathrm{Hom}_{\mathcal{C}}(x, \mathrm{colim}_{\mathcal{C}}(-))$ as retract of $\mathrm{Hom}_{\mathrm{Ind}(\mathcal{C})}(\mathrm{Ind}(r)(\hat{y}_{\mathcal{D}}(i(x))), -)$. By Yoneda, this shows that $\mathrm{Hom}_{\mathcal{C}}(x, \mathrm{colim}_{\mathcal{C}}(-))$ is itself corepresentable, by a retract of $\mathrm{Ind}(r)(\hat{y}_{\mathcal{D}}(i(x)))$, and so we have that $\mathrm{colim}_{\mathcal{C}}$ admits a left adjoint.

Assume now that $\mathcal{C}$ is stable and let us prove the equivalence $(2) \iff (3)$. The implication $(2) \Rightarrow (3)$ it's obvious: since retracts of dualizable objects are dualizable in any closed symmetric monoidal category, it suffices to remember that in Remark 1.5 we showed that compactly generated categories are dualizable in $\mathrm{Pr}^{\mathrm{L}}_{\mathrm{st}}$. Consider $(3) \Rightarrow (2)$. Let $\mathcal{C}^{\vee}$ the dual object of $\mathcal{C}$ in $\mathrm{Pr}^{\mathrm{L}}_{\mathrm{st}}$. Since $\mathcal{C}$ is κ-accessible for some regular cardinal

κ, we can construct a Bousfield localization

$$\mathrm{Ind}(\mathcal{C}^{\kappa}) \underset{\mathrm{incl}}{\overset{l}{\rightleftarrows}} \mathcal{C} \quad (\perp)$$

where $\mathcal{C}^{\kappa}$ denotes the full subcategory of $\mathcal{C}$ spanned by the κ-compact objects. Since $\mathrm{Ind}(\mathcal{C}^{\kappa})$ is compactly generated, it suffices to produce a colimit-preserving functor $i : \mathrm{Ind}(\mathcal{C}^{\kappa}) \to \mathcal{C}$ such that $l \circ i \simeq \mathrm{id}_{\mathcal{C}}$. Notice that, a priori, the inclusion incl is not colimit preserving! Of course at the end it will be $i \simeq \mathrm{incl}$. Anyway, since tensoring with $\mathcal{C}^{\vee}$ is a 2-functor, we obtain a Bousfield localization

$$\mathrm{Ind}(\mathcal{C}^{\kappa}) \otimes \mathcal{C}^{\vee} \underset{\mathrm{incl}\otimes\mathrm{id}_{\mathcal{C}^{\vee}}}{\overset{l\otimes\mathrm{id}_{\mathcal{C}^{\vee}}}{\rightleftarrows}} \mathcal{C} \otimes \mathcal{C}^{\vee} \quad (\perp)$$

Apply now the 2-functor $\mathrm{Fun}^{L}(\mathrm{Sp}, -)$ to produce the commutative diagram

$$\begin{array}{ccc} \mathrm{Fun}^{L}(\mathrm{Sp}, \mathrm{Ind}(\mathcal{C}^{\kappa}) \otimes \mathcal{C}^{\vee}) & \simeq & \mathrm{Fun}^{L}(\mathcal{C}, \mathrm{Ind}(\mathcal{C}^{\kappa})) \\ \downarrow{\scriptstyle (l\otimes\mathrm{id}_{\mathcal{C}^{\vee}})\circ -} & & \downarrow{\scriptstyle l\circ -} \\ \mathrm{Fun}^{L}(\mathrm{Sp}, \mathcal{C} \otimes \mathcal{C}^{\vee}) & \simeq & \mathrm{Fun}^{L}(\mathcal{C}, \mathcal{C}) \end{array}$$

Here the equivalences are provided by the duality datum. Lift now the identity functor from the right lower corner: via the horizontal bottom equivalence it corresponds to the coevaluation $c : \mathrm{Sp} \to \mathcal{C} \otimes \mathcal{C}^{\vee}$, and since mapping out of spectra consists in picking an object, this map lifts through the left vertical morphisms, and hence it provides a colimit preserving functor $i : \mathcal{C} \to \mathrm{Ind}(\mathcal{C}^{\kappa})$ such that $l \circ i \simeq \mathrm{id}_{\mathcal{C}}$. □

Example 1.8. Let us give some examples for the sake of clarity.

(1) Let $R \in \mathrm{CAlg}(\mathrm{Sp})$ be an $\mathbb{E}_{\infty}$-ring. Then the $(\infty, 1)$-category $\mathrm{Mod}_{R}$of R-module spectra is compactly generated. Moreover, when equipped with the tensor product of R-module spectra, it becomes a *rigidly-compactly generated* $(\infty, 1)$-category (that is, compactly generated $(\infty, 1)$-categories for which the dualizable and compact objects coincide and the tensor product preserves compact objects).

(2) Let X be a quasi-compact quasi-separated scheme and let $\mathrm{QCoh}(X)$ the derived stable $(\infty, 1)$-category of quasi-coherent sheaves. This can be constructed via the limit

$$\mathrm{QCoh}(X) = \lim_{\mathrm{Spec}(A)\to X} \mathrm{Mod}_{HA}$$

where the diagram is given by writing X as a colimit of its open affine pieces. Again, this is a rigidly-compactly generated $(\infty, 1)$-category: the compact objects coincide with the dualizable ones (which are identified with the perfect ones). Notice that this is no longer true if X is quasi-compact quasi-separated stack. We need some perfectness assumption: see [BZFN10].

(3) Let X be a locally compact Hausdorff space. We claim that $\mathrm{Shv}(X; \mathrm{Sp})$ is dualizable but not compactly-generated[3]. One possible proof of dualizability uses Lurie's covariant Verdier duality theorem [Lur17,

[3] Compactly-generation is hard and we refer the reader to [Har23, Proposition 3.4]. There it is proven, under hypercompleteness assumption, that $\mathrm{Shv}(X; \mathrm{Sp})$ is compactly generated if and only if X is totally disconnected. We would like to point out that this generalizes an argument of Neeman.

Theorem 5.5.5.1]; in the next section we give another proof.

2 Compactness

The goal of this section is to prove another equivalent characterisation to being compactly assembled. Indeed, we may argue that Theorem 1.7 is somehow unsatisfactory, since it characterise compactly assembled $(\infty, 1)$-categories *externally*. To correct this problem, we introduce the concept of compact maps: they will allow us to manipulate compactly assembled categories as compact objects allow us to manipulate compactly generated categories.

Definition 2.1. Let $\mathcal{M}$ be an $(\infty, 1)$-category with filtered colimits and let $f : x \to y$ be a morphism in $\mathcal{M}$. We will say that f is a *compact morphism* if for any filtered diagram $z_\bullet : I \to \mathcal{M}$ there exists a diagonal filler in the following commutative square:

$$\begin{array}{ccc} \operatorname{colim}_I \operatorname{Hom}(y, z_i) & \longrightarrow & \operatorname{Hom}(y, \operatorname{colim}_I z_i) \\ {\scriptstyle -\circ f}\downarrow & \swarrow & \downarrow{\scriptstyle -\circ f} \\ \operatorname{colim}_I \operatorname{Hom}(x, z_i) & \longrightarrow & \operatorname{Hom}(x, \operatorname{colim}_I z_i) \end{array}$$

Here the horizontal maps are induced by the universal property of colimits.

We have the following slogan: the definition of compact maps is supposed to be a way of encoding, using only the map $f : x \to y$, the property that it has a "compact image", or perhaps factors through a compact object, even in the absence of compact objects.

Example 2.2. Clearly, an object $x \in \mathcal{M}$ is compact if and only if id_x is a compact map.

Definition 2.3. Let $\mathcal{M}$ be an $(\infty, 1)$-category with filtered colimits. Let $x_\bullet : \mathbb{N} \to \mathcal{N}$ be a diagram and x be its colimit. We will say that $x_\bullet$ is a *compact exhaustion of* x if each transition map $x_n \to x_{n+1}$ is compact. Given $y \in \mathcal{M}$, we will say that y is *compactly exhaustible* if there exists a compact exhaustion of it.

The following result is the generalisation we were discussing before.

Proposition 2.4 ([Ram24a, Theorem 2.39])**.** Let $\mathcal{M}$ be an $(\infty, 1)$-category with filtered colimits. Then $\mathcal{M}$ is compactly assembled if and only if it is generated under filtered colimits by compactly exhaustible objects.

Proof. The proof is not too complicated, but we don't have time. □

Example 2.5 (Continuation of Example 1.8, point (3))**.** Let X be a locally compact Hausdorff space. We claimed before that $\mathrm{Shv}(X; \mathrm{Sp})$ is dualizable but not compactly-generated. We now claim that it is compactly assembled. We refer the reader to [AK24, Proposition 2.2.20]: there it is proved that X is locally compact if and only if $\mathrm{Shv}(X) = \mathrm{Shv}(X; \mathrm{Spc})$ is compactly assembled.

Example 2.6 ([Sch24, Proposition 2.9])**.** The 1-category of seminormed rings is compactly assembled. To prove that, let us denote by $\mathbb{Z}[T]_{\leq r}$ the free seminormed ring with an element T with $|T| \leq r$. We first claim that, for $s > r$ the map $\mathbb{Z}[T]_{\leq s} \to \mathbb{Z}[T]_{\leq r}$ is compact: we need to check that for any filtered colimit of seminormed rings $R = \operatorname{colim}_i R_i$ and an element $T \in R$ with $|T|_R \leq r$, there is some index i such that $|T|_{R_i} \leq s$; this follows from $|T|_R$ being the infimum of the $|T|_{R_i}$. It follows that $\mathbb{Z}[T]_{\leq r} = \operatorname{colim}_{s>r} \mathbb{Z}[T]_{\leq s}$ is compact exhaustible, and in particular Proposition 2.4 implies that the 1-category of seminormed rings is compactly assembled (since it is generated under colimits by this example by varying $r \in \mathbb{R}_{>0}$).

3 Local rigidity and rigidity

As we observed in Example 1.8, many stable presentable $(\infty,1)$-categories appearing in nature are not just compactly generated (and hence dualizable over Sp), they are also equipped with a symmetric monoidal structure for which the dualizable objects coincide with the compact ones and for which the tensor product preserves compact objects: they are *rigidly-compactly generated*. Since many does not mean all, we would like to develop a theory that allows us to treat also the case where our categories are "rigidly-compactly generated without being quite rigidly-compactly generated". To be precise, we are interested in a theory that:

(1) Allows us to study the case where we don't have enough compact objects to generate.

(2) Allows us to work over more general bases.

Rigid and, more generally, *locally rigid* $(\infty,1)$*-categories* achieve these goals and are what we want to discuss now.

Let us first recall the definition of internal left adjoint.

Definition 3.1. Let $\mathcal{V} \in \mathrm{CAlg}(\mathrm{Pr}^{\mathrm{L}})$ be a presentably symmetric monoidal $(\infty,1)$-category and let $f^{\mathrm{L}} : \mathcal{M} \to \mathcal{N}$ be in $\mathrm{Mod}_{\mathcal{V}}(\mathrm{Pr}^{\mathrm{L}})$ with right adjoint f^{R}. We will say that f^{L} is an internal left adjoint in $\mathrm{Mod}_{\mathcal{V}}(\mathrm{Pr}^{\mathrm{L}})$ if the square

$$\begin{array}{ccc} \mathcal{V}\otimes\mathcal{M} & \xrightarrow{\mathrm{id}_{\mathcal{V}}\otimes f^{\mathrm{L}}} & \mathcal{V}\otimes\mathcal{N} \\ \downarrow & & \downarrow \\ \mathcal{M} & \xrightarrow[f^{\mathrm{L}}]{} & \mathcal{N} \end{array}$$

is horizontally right adjointable and the right adjoint f^{R} is colimit-preserving.

In other terms, $f^{\mathrm{L}} : \mathcal{M} \to \mathcal{N}$ is an internal left adjoint in $\mathrm{Mod}_{\mathcal{V}}(\mathrm{Pr}^{\mathrm{L}})$ if the right adjoint f^{R} preserves colimits and the canonical projection map

$$v \otimes_M f^{\mathrm{R}}(n) \to f^{\mathrm{R}}(v \otimes_N n)$$

is an equivalence for all $v \in \mathcal{V}$ and $n \in \mathcal{N}$. That is, if f^{R} induces a right adjoint $\mathcal{V}$-module map $f^{\mathrm{R}} : \mathcal{N} \to \mathcal{M}$.

Example 3.2. Let's understand the case where $\mathcal{V}$ is the $(\infty,1)$-category of spectra Sp. Then a morphism $f^{\mathrm{L}} : \mathcal{M} \to \mathcal{N}$ in $\mathrm{Mod}_{\mathrm{Sp}}(\mathrm{Pr}^{\mathrm{L}})$ is an internal left adjoint if and only if it the right adjoint f^{R} preserves colimits. Furthermore, in the case where $\mathcal{M}$ is compactly generated, f^{L} is an internal left adjoint if and only if it preserves compact objects[4]. The first claim is [Ram24b, Corollary 3.8] and the second one is [Lur09, Proposition 5.5.7.2].

Definition 3.3. Let $f : \mathcal{B} \to \mathcal{C}$ be a morphism in $\mathrm{CAlg}(\mathrm{Pr}^{\mathrm{L}})$. We will say that $\mathcal{C}$ is *locally rigid over* $\mathcal{B}$ if:

(1) The multiplication map $\mathcal{C} \otimes_{\mathcal{B}} \mathcal{C} \to \mathcal{C}$ is an internal left adjoint in $\mathrm{Mod}_{\mathcal{C}\otimes_{\mathcal{B}}\mathcal{C}}(\mathrm{Pr}^{\mathrm{L}})$.

(2) The $\mathcal{C}$ is dualizable over $\mathcal{B}$.

In other terms, $\mathcal{C}$ is locally rigid over $\mathcal{B}$ if it is dualizable over $\mathcal{B}$ in the sense of Remark 1.4 and the right adjoint $\mathcal{C} \to \mathcal{C} \otimes_{\mathcal{B}} \mathcal{C}$ to the multiplication map preserves colimits and is $(\mathcal{C} \otimes_{\mathcal{B}} \mathcal{C})$-linear.

[4]More generally, any internal left adjoint $f^{\mathrm{L}} : \mathcal{M} \to \mathcal{N}$ preserves atomic objects. If $\mathcal{M}$ is atomically generated, the converse holds: if f^{L} preserves atomic objects, then f^{L} is an internal left adjoint.

We will see in Example 3.9 that a compactly generated $(\infty, 1)$-category is locally rigid over Sp if and only if every compact object is dualizable and the multiplication map preserves compact objects. To properly get back the notion of rigidly-compactly generated categories we still need to impose that "dualizable objects are compact". For this we need the notion of atomic object.

Definition 3.4. Let $\mathcal{V} \in \mathrm{CAlg}(\mathrm{Pr}^{\mathrm{L}})$ be a presentably symmetric monoidal $(\infty, 1)$-category and let $\mathcal{M} \in \mathrm{Mod}_{\mathcal{V}}(\mathrm{Pr}^{\mathrm{L}})$ be a $\mathcal{V}$-module. An object $x \in \mathcal{M}$ is called $\mathcal{V}$-*atomic* if the $\mathcal{V}$-linear functor $- \otimes x : \mathcal{V} \to \mathcal{M}$ is an internal left adjoint.

In other words, and object $x \in \mathcal{M}$ is $\mathcal{V}$-atomic if the right adjoint $\mathrm{map}_{\mathcal{M}}(x, -) : \mathcal{M} \to \mathcal{V}$ preserves colimits and the canonical map

$$v \otimes_{\mathcal{V}} \mathrm{map}_{\mathcal{M}}(x, y) \to \mathrm{map}_{\mathcal{M}}(x, v \otimes y)$$

is an equivalence for all $v \in \mathcal{V}$ and $y \in \mathcal{M}$.

Example 3.5. Let $\mathcal{V} \in \mathrm{CAlg}(\mathrm{Pr}^{\mathrm{L}})$ be a presentably symmetric monoidal $(\infty, 1)$-category. Then $\mathcal{V}$-atomic objects in $\mathcal{V}$ are exactly the dualizable objects. Indeed, if $x \in \mathcal{V}$ is dualizable with dual $x^{\vee}$, then we have an adjunction $- \otimes x \dashv - \otimes x^{\vee}$ that identifies $\underline{\mathrm{Hom}}_{\mathcal{V}}(x, -) \simeq - \otimes x^{\vee}$ with a colimit preserving and $\mathcal{V}$-linear functor. Conversely, if x is $\mathcal{V}$-atomic, then the dual can be identified with $x^{\vee} \simeq \underline{\mathrm{Hom}}_{\mathcal{V}}(x, 1)$.

Example 3.6. It is easy that Sp-atomic objects are exactly compact objects in the usual sense. Let $\mathcal{M}$ be a stable presentable $(\infty, 1)$-category and $x \in \mathcal{M}$ and consider $- \otimes x : \mathrm{Sp} \to \mathcal{M}$. Accordingly to Example 3.2, the functor is an internal left adjoint if and only if its right adjoint $\mathrm{map}_{\mathcal{M}}(x, -)$ preserves colimits, which happens if and only if x is compact, as one can show by applying $\Omega^{\infty+n}$ for every integer n.

The next result shows that atomic objects behave similarly to compact ones.

Lemma 3.7 ([Ram24a, Corollary 1.26])**.** Let $\mathcal{V} \in \mathrm{CAlg}(\mathrm{Pr}^{\mathrm{L}})$ be a presentably symmetric monoidal $(\infty, 1)$-category and let $\mathcal{M} \in \mathrm{Mod}_{\mathcal{V}}(\mathrm{Pr}^{\mathrm{L}})$ a $\mathcal{V}$-module. The full subcategory $\mathcal{M}_{\mathrm{at}}$ of atomic objects of $\mathcal{M}$ is essentially small. In fact, if the unit $1_{\mathcal{V}}$ is κ-compact, we have $\mathcal{M}_{\mathrm{at}} \subseteq \mathcal{M}^{\kappa}$.

We can finally strengthen the definition of locally rigid algebra.

Definition 3.8. Let $f : \mathcal{B} \to \mathcal{C}$ be a morphism in $\mathrm{CAlg}(\mathrm{Pr}^{\mathrm{L}})$. We will say that $\mathcal{C}$ is *rigid over* $\mathcal{B}$ if:

(1) It is locally rigid over $\mathcal{B}$.

(2) The unit of $\mathcal{C}$ is $\mathcal{B}$-atomic.

Example 3.9. Let $\mathcal{C}$ be a compactly generated $(\infty, 1)$-category. Then $\mathcal{C}$ is Sp-rigid if and only if it is rigidly-compactly generated. Assume that $\mathcal{C}$ is rigid. Since the unit is Sp-atomic, it is compact. Hence every dualizable object is compact. For the converse, notice that, if $x \in \mathcal{C}$ is compact, the functor $- \otimes x : \mathcal{C} \to \mathcal{C}$ admits

$$\mathcal{C} \to \mathcal{C} \otimes \mathcal{C} \xrightarrow{\mathrm{id} \otimes \mathrm{map}_{\mathcal{C}}(x, -)} \mathcal{C} \otimes \mathrm{Sp} \simeq \mathcal{C}$$

as right adjoint. Finally, since we are over spectra, the multiplication map is an internal left adjoint if and only if it its right adjoint preserves colimits. Since $\mathcal{C}$ and $\mathcal{C} \otimes \mathcal{C}$ are compactly generated, this is the same of the multiplication preserving compact objects! Conversely, if $\mathcal{C}$ is rigidly-compactly generated, then it is locally rigid (since it is dualizable by Remark 1.5 and the multiplication map is an internal left adjoint, since it reduces to the tensor product, which preserves compact objects since they coincide with the dualizable ones). It is also rigid, since the unit, being dualizable, is compact, hence Sp-atomic by Example 3.6.

4 Scholze's results

We begin with an easy result that will simplify Scholze's proofs.

Lemma 4.1. Let $\mathcal{V} \in \mathrm{CAlg}(\mathrm{Pr}^L)$ be a presentably symmetric monoidal $(\infty, 1)$-category and let $f^L : \mathcal{M} \to \mathcal{N}$ be in $\mathrm{Mod}_{\mathcal{V}}(\mathrm{Pr}^L)$.

(1) Assume that $\mathcal{M}$ is atomically generated, that is it is generated under $\mathcal{V}$-colimits by atomic objects, and that f^L preserves them. Then f^L is an internal left adjoint

(2) Assume that $\mathcal{V}$ is generated under colimits by its dualizable objects. If f^R is colimit preserving, then f^L is an internal left adjoint.

Proof. Point (1) is [Ram24a, Corollary 1.31]: the idea is that being $\mathcal{V}$-atomically generated amounts to the statement that the various $- \otimes m : \mathcal{V} \to \mathcal{M}$, for $m \in \mathcal{M}_{at}$, are internal left adjoints and have jointly conservative right adjoints. Indeed, by [Ram24a, Lemma 1.30] we deduce that f^L is an internal left adjoint if and only if each $f^L(- \otimes m) \simeq - \otimes f^L(m)$ is; this is in our assumptions since f^L preserves atomic objects.

Point (2), once we know that f^R is colimit-preserving, follows from [Ram24b, Corollary 3.8]: it shows that for every dualizable object $v \in \mathcal{V}$ the natural map $v \otimes f^R(n) \to f^R(v \otimes n)$ is an equivalence for every $n \in \mathcal{N}$. Since $\mathcal{V}$ is generated by dualizable objects under colimits and both sides preserves them, we are done. □

We also need to understand the behaviour of dualizable categories under filtered colimits.

Lemma 4.2 ([Efi25, Proposition 1.72])**.** Let $(\mathcal{C}_i)_i$ be a filtered diagram of dualizable stable $(\infty, 1)$-categories, with colimit-preserving functors whose right adjoints also commute with colimits. Then.

(1) The colimit $\mathrm{colim}\, \mathcal{C}_i$ is dualizable.

(2) Passing to compact objects, the natural functor $\mathrm{colim}(\mathcal{C}_i)_c \to (\mathrm{colim}\, \mathcal{C}_i)_c$ of small stable $(\infty, 1)$-categories is an equivalence.

Proof. We don't give a detail proof of this result, but rather we refer the reader to [Efi25, Proposition 1.65] for the first claim and to [Efi25, Proposition 1.71] for the second. □

We can finally prove Scholze's results.

Proposition 4.3 ([Sch24, Proposition 10.1])**.** Let C be an algebraically closed Banach field. Then $\mathcal{D}_{mot}(C)$ is rigidly-compactly generated. Moreover, if C has equal characteristic and we pick a splitting $k \to C$ of its residue field k, then a generating family of compact objects is given by $\mathbb{Z}_{mot}[X](-j)$ where X is a smooth projective variety over k, and $j \geq 0$.

Proof. We don't have time for the proof. We just remark that one can reduce to the case where C is of equal characteristics, notice that $\mathbb{Z}_{mot}[Y](-j)$ is dualizable when Y is smooth projective over C, that the unit is compact (so that all dualizable objects are compact) and then show that $\mathbb{Z}_{mot}[X_C](-j)$ generates $\mathcal{D}_{mot}(C)$. Since $\mathbb{Z}$ generates the torsion part, it is sufficient to deal with rational coefficients.

We need to show that, if $M \in \mathcal{D}_{mot}(C)$ is such that all the maps $\mathbb{Z}_{mot}[X_C](-j) \to M$ are zero for all such X and j, then $M = 0$. Since it suffices to show that $M(C') = 0$ for all complete algebraically closed extensions C'/C, one can argue on the topological degree of the extension. In particular one reduces the proof to study the case where C_d/C is some extension of topological transcendence degree d. By picking some extension $C_{d-1} \subsetneq C_d$ of topological transcendence degree $d-1$ such that C_d/C_{d-1} is of transcendence degree 1, one reduces the claim to check the latter extension. The proof then follows by noting that the extension C_d/C_{d-1} corresponds to one of the types of points (2), (3) or (4) on the Berkovich line over C_{d-1}. $\square$

Proposition 4.4 ([Sch24, Proposition 10.3])**.** Let A be an analytic Banach ring of finite cohomological dimension. Then $\mathcal{D}_{\mathrm{mot}}(\mathcal{M}_{\mathrm{arc}}(A))$ is rigid.

More generally, let X be a qcqs arc-sheaf of finite cohomological dimension such that for each $x \in |X|$ there is some non-discrete algebraically closed Banach field $C(x)$ and a quasi-pro-étale map[5] $\mathcal{M}_{\mathrm{arc}}(C(x)) \to X$ with image x. Then $\mathcal{D}_{\mathrm{mot}}(X)$ is rigid over Sp.

Proof. We need to show that $\mathcal{D}_{\mathrm{mot}}(X)$ is locally rigid over Sp and that the unit of $\mathcal{D}_{\mathrm{mot}}(X)$ is Sp-atomic. This last fact is obvious from our assumptions: being Sp-atomic, accordingly to Example 3.6, consists in showing that the unit of $\mathcal{D}_{\mathrm{mot}}(X)$ is compact, but X is assumed to be of finite cohomological dimension! Notice, furthermore, that this implies that dualizable objects are compact.

Local rigidity over Sp consists in showing that $\mathcal{D}_{\mathrm{mot}}(X)$ is dualizable over Sp and that the multiplication map $\mathcal{D}_{\mathrm{mot}}(X) \otimes \mathcal{D}_{\mathrm{mot}}(X) \to \mathcal{D}_{\mathrm{mot}}(X)$ is an internal left adjoint. Let's treat these two facts separately.

(1) Accordingly to Theorem 1.7, to show that $\mathcal{D}_{\mathrm{mot}}(X)$ is Sp-dualizable it suffices to show that it is compactly assembled. We now use Proposition 2.4: it suffices to show that $\mathcal{D}_{\mathrm{mot}}(X)$ is generated under filtered colimits by compactly exhaustible objects.

First of all, let us describe some of the compact maps. We claim that for an inclusion $U \subseteq V \subseteq \mathbb{A}^n_X$ open subsets that factors over a compact subset $K \subseteq \mathbb{A}^n_X$, that is $U \subseteq K \subseteq V$, the induced morphism $\mathbb{Z}_{\mathrm{mot}}[U] \to \mathbb{Z}_{\mathrm{mot}}[V]$ is compact: this follows since the induced map $\mathrm{Hom}(V, -) \to \mathrm{Hom}(U, -)$ factors trough $\mathrm{Hom}(K, -)$, which commutes with filtered colimits. Now, since any $V \subseteq \mathbb{A}^n_X$ can be written as a union of such U strictly contained in V , we see that $\mathbb{Z}_{\mathrm{mot}}[V]$ is compactly exhaustible, so that it remains to show that those objects generate $\mathcal{D}^{\mathrm{eff}}_{\mathrm{mot}}(X)$ (because their negative Tate twists generate the all of $\mathcal{D}_{\mathrm{mot}}(X)$).

Let $\mathcal{C} \subseteq \mathcal{D}^{\mathrm{eff}}_{\mathrm{mot}}(X)$ be the full subcategory spanned by the $\mathbb{Z}_{\mathrm{mot}}[V]$. Since the inclusion preserves compact maps and since $\mathcal{C}$ is compactly assembled, the right adjoint $\mathcal{D}^{\mathrm{eff}}_{\mathrm{mot}}(X) \to \mathcal{C}$ preserves colimits[6]. In particular, we can pass to stalks and reduce ourselves to prove the that $\mathcal{C} = \mathcal{D}^{\mathrm{eff}}_{\mathrm{mot}}(X)$ for $|X|$ a point. Then $X = \mathcal{M}_{\mathrm{arc}}(C)/G$ for some profinite group G acting continuously on an algebraically closed non-discrete Banach field C. We now use [Sch24, Proposition 5.13]: Sebastian showed that for an analytic ring A of finite cohomological dimension, $\mathcal{D}^{\mathrm{eff}}_{\mathrm{mot}}(\mathcal{M}_{\mathrm{arc}}(A))$ is generated by $\mathbb{Z}_{\mathrm{mot}}[V]$ for $V \subseteq \mathbb{A}^n_A$ open; one can runthe same proof in the case where a profinite group G acts continuously on an algebraically closed non-discrete Banach field C.

(2) We need to show that the multiplication map $\mathcal{D}_{\mathrm{mot}}(X) \otimes \mathcal{D}_{\mathrm{mot}}(X) \to \mathcal{D}_{\mathrm{mot}}(X)$ is an internal left adjoint in $\mathrm{Mod}_{\mathcal{D}_{\mathrm{mot}}(X) \otimes \mathcal{D}_{\mathrm{mot}}(X)}(\mathrm{Pr}^L)$. We wish to apply point (1) of Lemma 4.1 in the case where f^L is given by the multiplication map, so let us verify its assumptions[7]. That is, we need to show that

[5] Recall that the geometric points are given by algebraically closed Banach fields. The assumption of being non-discrete is used in the proof.

[6] Maybe a proof would be nice

[7] We are in the case where $\mathcal{V} = \mathcal{M} = \mathcal{D}_{\mathrm{mot}}(X) \otimes \mathcal{D}_{\mathrm{mot}}(X)$ and $f^L = - \otimes - : \mathcal{D}_{\mathrm{mot}}(X) \otimes \mathcal{D}_{\mathrm{mot}}(X) \to \mathcal{D}_{\mathrm{mot}}(X)$, so the assumptions we need to verify are: $\mathcal{V}$ is $\mathcal{V}$-atomically generated and f^L preserves atomic objects. However, Example 3.5 tells us that we just need to show that $\mathcal{V}$ is generated by dualizable objects.

$\mathcal{D}_{mot}(X) \otimes \mathcal{D}_{mot}(X)$ is generated under colimits by its dualizable objects and that the multiplication map preserves compact objects.

That the multiplication map preserves compact objects is a consequence of the previous point: we have showed that the tensor product preserves compact maps, and hence compact objects. Since to show that $\mathcal{D}_{mot}(X) \otimes \mathcal{D}_{mot}(X)$ is generated under colimits by its dualizable objects, it suffices to show that $\mathcal{D}_{mot}(X)$ is generated under colimits by its dualizable objects, Scholze reduces the proof in two cases.

In the first one, he still assumes that $X = \mathcal{M}_{arc}(C)/G$ is a point. By picking a cofinal system of open subgroups $G_i \subseteq G$, we see that

$$\mathcal{M}_{arc}(C) \simeq \lim \mathcal{M}_{arc}(C)/G_i$$

so that, by taking $\mathcal{D}_{mot}(-)$ on both sides we can apply Lemma 4.2 (the categories $\mathcal{D}_{mot}(\mathcal{M}_{arc}(C)/G_i)$ are dualizable by the previous point!): it implies that every compact object of $\mathcal{M}_{arc}(C)$ descends to some $\mathcal{M}_{arc}(C)/G_i$, as does its dual, as well as the duality maps; thus, it descends to a compact dualizable object. Here we have used Proposition 4.3 since we needed $\mathcal{D}_{mot}(\mathcal{M}_{arc}(C))$ to be rigidly-compactly generated. Now the claim follows since the pushforward along the finite étale map $\mathcal{M}_{arc}(C)/G_i \to \mathcal{M}_{arc}(C)/G$ preserves compact-dualizable objects, and these objects generate.

In the second case, one applies a similar argument and we just sketch it (since we didn't introduce the appropriate terminology). It suffices to see that there is a generating family of objects that are sequential colimits along trace-class maps (these are to dualizable objects as compact maps are to compact objects). Pick any $x \in X$ and write it as a cofiltered limit of closed neighborhoods $Z_i \subseteq X$. Using Lemma 4.2 again, the compact dualizable objects in the fibre at x spread to some Z_i. If $M \in \mathcal{D}_{mot}(Z_i)$ is compact dualizable and $U_i \subseteq Z_i$ is an open neighborhood of x in X, without loss of generality a countable union of closed subsets, then the extension by 0 from U_i to X of $M|_{U_i}$ is such a sequential colimit along trace-class maps, and these objects generate.

Putting all together, we conclude. ☐

Corollary 4.5 ([Sch24, Corollary 10.6])**.** Let $X \to S \leftarrow Y$ be a diagram of qcqs arc-stacks as before. Then the exterior tensor product defines a fully faithful functor

$$- \boxtimes - : \mathcal{D}_{mot}(X) \otimes_{\mathcal{D}_{mot}(S)} \mathcal{D}_{mot}(Y) \to \mathcal{D}_{mot}(X \times_S Y).$$

Proof. Before doing the proof, let us note that Scholze does not say anything explicitly on why, for a morphism $X \to S$, the induced pullback functor $\mathcal{D}_{mot}(S) \to \mathcal{D}_{mot}(X)$ exhibits $\mathcal{D}_{mot}(X)$ as rigid over $\mathcal{D}_{mot}(S)$. This should follow from a result by Arinkin–Gaitsgory–Kazhdan–Raskin–Rozenblyum–Varshavsky, [AK24, Theorem 4.3.1]: since $\mathcal{D}_{mot}(S)$ is locally rigid over Sp , $\mathcal{D}_{mot}(X)$ is locally rigid over $\mathcal{D}_{mot}(S)$ if and only if its locally rigid over Sp, so that we have nothing to prove. Confession: I've not understood why the unit of $\mathcal{D}_{mot}(X)$ is $\mathcal{D}_{mot}(S)$-atomic. Hopefully someone has an idea... Anyway, let R be the right adjoint of $\boxtimes$. We need to show that the unit map

$$\mathrm{id}_{\mathcal{D}_{mot}(X) \otimes_{\mathcal{D}_{mot}(S)} \mathcal{D}_{mot}(Y)} \to R(- \boxtimes -)$$

is an equivalence. But since R is linear over $\mathcal{D}_{mot}(S)$ and colimit preserving, we can reduce ourselves to check that this is true when the input is given by the unit of $\mathcal{D}_{mot}(X) \otimes_{\mathcal{D}_{mot}(S)} \mathcal{D}_{mot}(Y)$. This is obvious, since the projection formula implies the Kunneth formula. ☐

References

[AK24] Phil P utzstu ck Achim Krause, Thomas Nikolaus. *Sheaves on Manifolds*. Unpublished, July 13 2024. Available online at Sheaves on Manifolds. 3, 4, 6, 11

[BZFN10] David Ben-Zvi, John Francis, and David Nadler. Integral transforms and drinfeld centers in derived algebraic geometry. *Journal of the American Mathematical Society*, 23(4):909–966, 2010. 5

[Efi25] Alexander I. Efimov. K-theory and localizing invariants of large categories, 2025. 9

[Har23] Oscar Harr. Compact sheaves on a locally compact space, 2023. 5

[Lur09] Jacob Lurie. *Higher Topos Theory (AM-170)*. Princeton University Press, 2009. Available online at Higher Topos Theory. 7

[Lur17] Jacob Lurie. *Higher Algebra*. Unpublished, September 2017. Available online at Higher Algebra. 1, 6

[Lur18] Jacob Lurie. *Spectral Algebraic Geometry*. Unpublished, 2018. Available online at Spectral Algebraic Geometry. 3

[Ram24a] Maxime Ramzi. Dualizable presentable ∞-categories, 2024. Available online at Dualizable presentable $(\infty, 1)$-categories. 6, 8, 9

[Ram24b] Maxime Ramzi. Locally rigid ∞-categories, 2024. 7, 9

[Sch24] Peter Scholze. Berkovich motives, 2024. 6, 9, 10, 11