HIOB-SEMINAR SOSE25 – BERKOVICH MOTIVES

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Mondays, 12h-14h, SFB Lecture Hall

Overview

In order to prove the Weil conjectures, relating the geometry of complex varieties with the number of points of algebraic varieties over finite fields, Grothendieck introduced the theory of ℓ-adic étale cohomology of algebraic varieties. Inspired by the underlying conjectural philosophy of motives, Voevodsky then associated to each algebraic variety X a $\mathbb{Z}$-linear derived category of motives $\mathrm{DM}(X)$, whose ℓ-adic part is related to the ℓ-adic étale cohomology of X, and whose rational part is related to the algebraic K-theory of X [Voe00]. Voevodsky's programme culminated with the proof of the Bloch–Kato conjecture, relating the étale cohomology of a field with its Milnor K-theory [Voe11, Rio14].

In parallel to this algebraic story, Tate initiated in [Tat71] the development of rigid-analytic geometry, as a non-archimedean analogue of complex analytic geometry. Variants of this theory were further developed by Raynaud [Ray74] and Huber [Hub96], and the étale cohomology of these rigid-analytic spaces found applications in several related areas, typically in the proof by Harris–Taylor of the local Langlands correspondence for GL_n [HT01]. A subsequent motivic theory for rigid-analytic varieties, similar to the algebraic theory of Voevodsky, was then initiated by Ayoub [Ayo15], and further developed by Ayoub–Gallauer–Vezzani [AGV22] and Binda–Gallauer–Vezzani [BGV23].

The theory of Berkovich spaces [Ber90], and of their étale cohomology [Ber93], provides a common framework for algebraic, complex analytic, and rigid-analytic geometries. The goal of this seminar is to study the recent paper [Sch24], where Scholze constructs a theory of étale motives for arbitrary Berkovich spaces. This theory satisfies good categorical properties, and recovers the étale version of Voevodsky's theory over a discrete field, the theory of Betti sheaves over $\mathbb{C}$, and is closely related to Ayoub's theory over a non-archimedean field. To motivate the constructions of this paper, we will cover some of the relevant background material on analytic geometry (see for instance [Con07]) and on motivic categories (see for instance [MVW06]).

Talks

Talk 1: Introduction (Tess Bouis, 28.04.2025). Overview of the seminar and distribution of the talks. Please attend if you are considering giving a talk.

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Talk 2: Banach rings (Antoine Sedillot, 05.05.2025). Mainly follow [Sch24, §2]. The goal of this talk is to give intuitions about the basics of Berkovich geometry. Define the notions of a Banach ring [Sch24, Definitions 2.1 and 2.2], of the Berkovich spectrum of a Banach ring [Sch24, Definition 2.13]. Discuss (and in particular compare the algebraic and analytic variants in) the examples of Banach fields, of the integers $\mathbb{Z}$, of the free Banach algebras over a Banach ring, of the ball, of $\mathbb{A}^1$, and of $\mathbb{P}^1$. Several introductury texts are available online; see for instance [Bak08, §1] for the last example.

Talk 3: The arc-topology (12.05.2025). Begin with a quick reminder on Grothendieck topologies, following for instance [Kha23, §2]. Briefly introduce Bhatt–Mathew's definition of the (algebraic) arc-topology [BM21, Definition 1.1] and explain the motivation for this topology via the example of étale cohomology [BM21, Theorem 1.8]. Then introduce Scholze's (analytic) arc-topology in the setting of Banach rings [Sch24, Definition 3.1] and explain quickly why this defines a Grothendieck topology on (the opposite of) the category of Banach rings [Sch24, Propositions 3.2 and 3.3]. Explain carefully the fact that strictly totally disconnected Banach rings form a basis for the arc-topology [Sch24, Definition 3.10 to Theorem 3.13], and how the situation specializes when working over $\mathbb{C}$ [Sch24, Example 3.8 and Proposition 3.9].

Talk 4: Voevodsky/Ayoub's étale motives (Marc Hoyois, 19.05.2025). This talk is here to motivate what will happen in the next talks. Introduce Voevodsky/Ayoub's category of étale motives associated to a scheme X, and discuss its basic properties, following [Cis21, §1] or any other resource. Also state the cancellation theorem.

Talk 5: Definition of Berkovich motives (26.05.2025). Define small arc-stacks, and compare this to the definition of Berkovich spaces, as introduced for instance in [LP24]. Introduce the category of Berkovich motives $\mathcal{D}_{\mathrm{mot}}(X)$ associated to a small arc-stack X [Sch24, Definitions 4.10, 5.1, 5.2, 5.18, and 9.1]. State the cancellation theorem [Sch24, Theorem 1.9], and comment on how this allows to understand more concretely the category $\mathcal{D}_{\mathrm{mot}}(X)$. Discuss some of the properties of finitary arc-sheaves from [Sch24, §4].

Talk 6: Ball-invariant arc-sheaves (02.06.2025). Discuss some of the properties of ball-invariant finitary arc-sheaves from [Sch24, §5]. Compare the notions of $\mathbb{A}^1$-localisation and $\mathbb{B}$-localisation (see for instance [Sch24, Remark 5.7] and [KST19, §1.2]). Then discuss some of the categorical properties of Berkovich motives following [Sch24, §9].

Talk 7: Free motivic sheaves (16.06.2025). Introduce and discuss the properties of the free motivic sheaves, following [Sch24, §5.1 and §6]. Discuss in particular the comparison of Berkovich motives with finite coefficients with ℓ-adic étale sheaves [Sch24, Theorem 6.7].

Talk 8: The cancellation theorem (23.06.2025). State and prove the cancellation theorem [Sch24, Theorem 7.1]. Compare to the statements and proofs in the algebraic and rigid-analytic settings (see [Voe10, Bac21] and [Ayo15, Vez17], respectively).

Talk 9: The arc-local K-theory (30.06.2025). Recall the definition of connective K-theory of commutative rings. Say a word about how algebraic K-theory appears in Voevodsky's theory of motives. Define the arc-local K-theory $\overline{K}(R)$ of a Banach ring R [Sch24, Definition 8.3 and Proposition 8.4], and explain the rational Adams decomposition [Sch24, Corollary 8.6]. Finally, state the second part of [Sch24, Theorem 8.13], and explain as much as possible from its proof.

Talk 10: Rigid categories (07.07.2025). Give an introduction to the notions of compactly generated category and of rigid category, as discussed in [KNP24, §4.3 and §4.4]. Talk in particular about the categorical Künneth formula [Sch24, Corollary 10.6], state the theorem that the category of Berkovich motives $\mathcal{D}_{\mathrm{mot}}(X)$ is rigid under mild assumptions on X [Sch24, Proposition 10.3], and discuss its proof.

Talk 11: Comparison with Voevodsky/Ayoub's étale motives (14.07.2025). Prove that over a discrete field k, Berkovich motives are equivalent to Voevodsky/Ayoub's étale motivic sheaves [Sch24, Theorem 11.1]. Then explain the description of $\mathcal{D}_{\mathrm{mot}}(C)$, for C the completed algebraic closure of $k((T))_{1/2}$, in terms of motivic nearby cycles [Sch24, Corollary 11.10], and how this could be used to give a similar description of $\mathcal{D}_{\mathrm{mot}}(\mathbb{C}_p)$ using [Sch24, Proposition 6.8] (see also [BGV23]).

References

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