Group theory for physicists
Problem set 13 (for the exercises in the week of Jan. 26)

For problems 3 and 4 you need to wait for the lecture on Jan. 20.

Problem 1   Totally antisymmetric tensors

Show that in the tensor space V_m^n there is no totally antisymmetric tensor of rank $n > m$.

Problem 2   Irreducibility under Gl(2)

a) Consider the example from Sec. 8.3.3 of the lecture and show that the subspace $T'_\kappa(1) \equiv \{e_\kappa|\alpha\rangle\,;\,|\alpha\rangle \in V_2^3\}$ is two-dimensional and is spanned by the tensors $|\kappa, 1, 1\rangle$ and $|\kappa, 2, 1\rangle$ of symmetry Θ_κ. Here, e_κ is the Young operator corresponding to the normal Young tableau $\Theta_\kappa = \begin{array}{|c|c|} \hline 1 & 2 \\ \hline 3 \\ \cline{1-1} \end{array}$.

b) Show that the subspace $T'_\kappa(1)$ is irreducible under Gl(2).

Problem 3   Pseudoreality of SU(2)

Show that for SU(2) the irreps 2 and $\bar{2}$ are equivalent.
Hint: Consider first the matrix

$$C = i\sigma_2 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

and show that $C\vec{\sigma}C^{-1} = -\vec{\sigma}^*$. Then show that for an arbitrary element $U = \exp(-\frac{i}{2}\vec{\sigma} \cdot \vec{\alpha})$ of SU(2) we have $CUC^{-1} = U^*$.

Problem 4   Equivalence of irreps of SU(m)

a) Show that for Gl(m) the Young diagram with 1 column and m rows corresponds to the 1-dimensional irrep of $g \in$ Gl(m) given by $g \to \det(g)$.

b) Show that for SU(m) this is the trivial representation.

c) Consider irreps of Gl(m) given by the Young diagrams $\Theta = (\lambda_1, \ldots, \lambda_m)$ and $\Theta' = (\lambda_1 + 1, \ldots, \lambda_m + 1)$. E.g., for $n = 11$, $m = 5$, and $\Theta = (6, 3, 2, 0, 0)$ these are the diagrams

$$\Theta = \quad \text{and} \quad \Theta' =$$

Show that $\Theta' = \begin{array}{|c|} \hline \\ \hline \\ \hline \\ \hline \end{array} \otimes \Theta$, where the first diagram on the RHS has m rows.

d) Show that for SU(m) Θ' and Θ are equivalent.

e) Show that for SU(m) Θ' and Θ are also equivalent if Θ' contains k additional columns with m boxes each.