Universität Regensburg, Institut für Theoretische Physik
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Group theory for physicists
Problem set 12 (for the exercises in the week of Jan. 19)

Note that the first problem is quite long, while the other two problems are short.

Problem 1 Transformation properties of vectors and tensors

In this problem we use the notation introduced in Sec. 7.3 of the lecture.

a) Show that a four-vector v^μ transforms in the $(\frac{1}{2}, \frac{1}{2})$ representation of the Lorentz group.

Hint: There are different ways to show this. E.g., consider how v^0 and $\vec{v}$ transform under rotations and note that $0 \oplus 1 = \frac{1}{2} \otimes \frac{1}{2}$. Or, note that the object $\zeta = \xi \otimes \eta$ (with spinor ξ and conjugate spinor η) transforms in $(\frac{1}{2}, \frac{1}{2})$, show that $\zeta \to A\zeta A^\dagger$ with $A \in \mathrm{Sl}(2, \mathbb{C})$, and use the homomorphism of Sec. 1.11 of the lecture.

b) Show that the trace of a rank-2 tensor is invariant under Lorentz transformations and thus transforms in the trivial representation $(0, 0)$.

c) Show that an antisymmetric rank-2 tensor transforms in the representation $(1, 0) \oplus (0, 1)$ of the Lorentz group.

Hint: Define a 6-dimensional basis $e_{[\mu\nu]} \equiv e_\mu \otimes e_\nu - e_\nu \otimes e_\mu$ for the space of antisymmetric rank-2 tensors in Minkowski space and study the action of the generators (which now correspond to case 1 in Sec. 6.5) on this basis.

d) Let $t^{\mu\nu}$ be a symmetric rank-2 tensor, i.e., $t^{\mu\nu} = t^{\nu\mu}$. Show that the traceless symmetric tensor $t^{\mu\nu} - \frac{1}{4} g^{\mu\nu} t^\lambda{}_\lambda$ transforms in the $(1, 1)$ representation of the Lorentz group.

Hint: This part simplifies if you assume the results of b) and c), which you are allowed to do even if you haven't managed to solve these parts. Also note $\frac{1}{2} \otimes \frac{1}{2} = 0 \oplus 1$.

Problem 2 Representation of the symmetric group on the tensor space

Show that the matrices

$$D(p)_{\{j\}\{i\}} = \delta_{j_{p_1} i_1} \cdots \delta_{j_{p_n} i_n} \qquad (p \in S_n)$$

introduced in Sec. 8.2 of the lecture form a representation of S_n on the tensor space V_m^n.

Problem 3 $\mathrm{Gl}(m)$ and S_n commute on the tensor space

Show that the representation matrices $D(g)$ ($g \in \mathrm{Gl}(m)$) commute with the representation matrices $D(p)$ ($p \in S_n$) on the tensor space V_m^n, i.e.,

$$pg|i\rangle_n = gp|i\rangle_n \quad \text{for all} \quad |i\rangle_n = |i_1 \cdots i_n\rangle \,.$$

Hint: Use the "important point #1" from Sec. 8.2 of the lecture.