Universität Regensburg, Institut für Theoretische Physik WS 2025/26
Prof. Dr. Tilo Wettig

Group theory for physicists

Problem set 11 (for the exercises in the week of Jan. 12)

Problem 1 Rotations about an axis

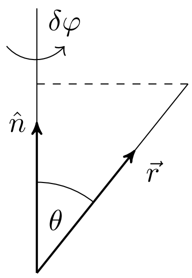
Show that the generator $J_{\hat{n}}$ of an SO(3) rotation about an arbitrary axis $\hat{n} = (n_1, n_2, n_3)$ with $|\hat{n}| = 1$ is given by

$$J_{\hat{n}} = \hat{n} \cdot \vec{J} = \sum_{k=1}^{3} n_k J_k \,,$$

where the J_k are the generators of SO(3) given by

$$J_1 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -i \\ 0 & i & 0 \end{pmatrix}, \qquad J_2 = \begin{pmatrix} 0 & 0 & i \\ 0 & 0 & 0 \\ -i & 0 & 0 \end{pmatrix}, \qquad J_3 = \begin{pmatrix} 0 & -i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

Hint: As shown in the figure, consider a rotation by a small angle $\delta\varphi$ and show that $\delta\vec{r} = \vec{r}\,' - \vec{r} = \hat{n} \times \vec{r}\,\delta\varphi$. Rewrite the cross product by means of the totally antisymmetric tensor ε and use $(J_k)_{ij} = -i\varepsilon_{ijk}$.



Problem 2 Rotations by the same angle are in the same class

Show that for any $R \in \mathrm{SO}(3)$ we have $RR_{\hat{n}}(\psi)R^{-1} = R_{\hat{n}'}(\psi)$ with $\hat{n}' = R\hat{n}$.

Problem 3 SU(2) matrices as exponentials of the generators

Show that for $\vec{\alpha} = \hat{n}\psi$ we have

$$e^{-i\frac{1}{2}\vec{\sigma}\cdot\vec{\alpha}} = \mathbb{1}\cos\frac{\psi}{2} - i\vec{\sigma}\cdot\hat{n}\sin\frac{\psi}{2}\,.$$

Here, $\hat{n}$ is a unit vector in three dimensions, and the vector $\vec{\sigma}$ contains the three Pauli matrices

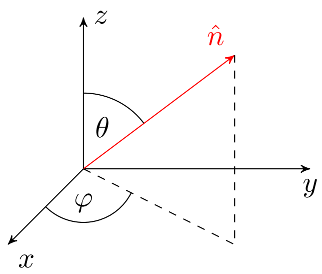
$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \qquad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

Hint: You can solve this problem without knowing anything about SU(2). First show $(\vec{\sigma}\cdot\hat{n})^2 = \mathbb{1}$.

Problem 4   Relationship between angle-and-axis parameters and Euler angles

In Sec. 6.7 we have discussed two different ways to parameterize rotations in three dimensions, i.e., angle-and-axis parameters and Euler angles. In the former case, we parameterize a rotation $R_{\hat{n}}(\psi)$ by the rotation angle ψ and the rotation axis $\hat{n}$ with

$$\hat{n} = \begin{pmatrix} \sin\theta\cos\varphi \\ \sin\theta\sin\varphi \\ \cos\theta \end{pmatrix}$$



The intervals of the parameters are $\psi, \theta \in [0, \pi]$ and $\varphi \in [0, 2\pi]$. In the latter case, we write a rotation as $R = R_z(\alpha)R_y(\beta)R_z(\gamma)$ with

$$R_y(\psi) = \begin{pmatrix} \cos\psi & 0 & \sin\psi \\ 0 & 1 & 0 \\ -\sin\psi & 0 & \cos\psi \end{pmatrix} \qquad \text{and} \qquad R_z(\psi) = \begin{pmatrix} \cos\psi & -\sin\psi & 0 \\ \sin\psi & \cos\psi & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

The intervals of the three Euler angles are $\alpha, \gamma \in [0, 2\pi]$ and $\beta \in [0, \pi]$. Prove the relationship between the two parameterizations,

$$\begin{aligned} \cos\psi &= 2\cos^2\frac{\beta}{2}\cos^2\frac{\alpha+\gamma}{2} - 1\,, \\ \varphi &= \left(\frac{1}{2}(\pi + \alpha - \gamma) + k\pi\right) \mod 2\pi\,, \\ \tan\theta &= (-1)^k \frac{\tan\frac{\beta}{2}}{\sin\frac{\alpha+\gamma}{2}}\,, \end{aligned}$$

where $k = 1$ if $(\alpha + \gamma) \mod 4\pi \in [\pi, 3\pi]$ and $k = 0$ otherwise.

Hints: To derive the first equation, note that all rotations by the same angle ψ are in the same class. Consider a rotation about the z-axis and compute the trace (i.e., character) of the rotation matrix for both parameterizations. To derive the remaining two equations, use the fact that $\hat{n}$ is left invariant by the rotation $R(\alpha, \beta, \gamma)$.

Remark: Since $R(\alpha, \beta, \gamma)$ is 2π-periodic in all three angles, we can shift the intervals of the angles. In particular, if we use $\gamma \in [-\pi - \alpha, \pi - \alpha]$, we always find $k = 0$.