

## Group theory for physicists

### Problem set 10 (for the exercises in the week of Dec. 22)

---

#### Problem 1 Representations and characters of $S_4$

- Draw all Young diagrams for  $S_4$  and determine the dimensions of the corresponding irreps. Check that  $\sum_i d_i^2 = 4!$  is satisfied.
- Compute the character table of  $S_4$  and check the orthogonality relations for the characters. Hint: Some irreps are associated to each other.

#### Problem 2 Representations and characters of $S_5$

Same as problem 1, but now for  $S_5$ .

#### Problem 3 Product representations of $S_5$

Consider the following product representations of  $S_5$  and find out which irreps of  $S_5$  occur in them (including the corresponding multiplicities). Use the characters from problem 2. The result of part a) is given for your convenience, but you should still prove it.

$$\begin{aligned}
 \text{a)} \quad & \begin{array}{|c|c|c|} \hline \square & \square & \square \\ \hline \square & & \end{array} \otimes \begin{array}{|c|c|c|} \hline \square & \square & \square \\ \hline \square & & \end{array} = \begin{array}{|c|c|c|c|c|} \hline \square & \square & \square & \square & \square \\ \hline \end{array} \oplus \begin{array}{|c|c|c|c|} \hline \square & \square & \square & \square \\ \hline \square & & & \end{array} \oplus \begin{array}{|c|c|c|} \hline \square & \square & \square \\ \hline \square & & \end{array} \oplus \begin{array}{|c|c|c|} \hline \square & \square & \square \\ \hline \square & & \end{array} \oplus \begin{array}{|c|c|c|} \hline \square & \square & \square \\ \hline \square & & \end{array} \oplus \begin{array}{|c|c|c|} \hline \square & \square & \square \\ \hline \square & & \end{array} \\
 \text{b)} \quad & \begin{array}{|c|c|c|} \hline \square & \square & \square \\ \hline \square & & \end{array} \otimes \begin{array}{|c|c|c|} \hline \square & \square & \square \\ \hline \square & & \end{array} = ? \\
 \text{c)} \quad & \begin{array}{|c|c|c|} \hline \square & \square & \square \\ \hline \square & & \end{array} \otimes \begin{array}{|c|c|c|} \hline \square & \square & \square \\ \hline \square & & \end{array} = ?
 \end{aligned}$$

Check the dimensions on both sides of the equations.

Hint: Try to find an efficient way of computing the multiplicities  $a_k$  by regarding the character table as a matrix and expressing  $a_k$  as a scalar product involving a row of the character table.

#### Problem 4 Baryon decuplet and octet

In the quark model baryons consist of three quarks. Quarks have several quantum numbers, e.g., the  $z$ -component  $I_3$  of isospin and the hypercharge  $Y = B + S$ , where  $B$  is the baryon number ( $B = \frac{1}{3}$  for all quarks) and  $S$  is the strangeness. We consider the following three states:  $\psi_1$  (up quark) with  $(I_3, Y) = (\frac{1}{2}, \frac{1}{3})$ ,  $\psi_2$  (down quark) with  $(I_3, Y) = (-\frac{1}{2}, \frac{1}{3})$ , and  $\psi_3$  (strange quark) with  $(I_3, Y) = (0, -\frac{2}{3})$ . For a product wave function  $\psi_i(1)\psi_j(2)\psi_k(3)$  involving three quarks,  $I_3$  and  $Y$  are given by the sums of the  $I_3$ - and  $Y$ -values of the three quarks.

- a) The functions  $|ijk\rangle \equiv \psi_i(1)\psi_j(2)\psi_k(3)$  with  $i, j, k = 1, \dots, 3$  furnish a 27-dimensional representation of  $S_3$ . Find out which irreps of  $S_3$  occur in this representation and the corresponding multiplicities. (The group action is given by  $(12)|ijk\rangle = |jik\rangle$  etc.)
- b) Find the 27 functions that transform in irreps of  $S_3$  and span the irreducible subspaces corresponding to these irreps. Give the  $(I_3, Y)$ -values of these functions. Use the techniques from Sec. 4.2 of the lecture.
- c) In an  $(I_3, Y)$ -diagram, draw points corresponding to the functions transforming in the irrep  $\square\square\square$ . In another  $(I_3, Y)$ -diagram, draw points corresponding to the functions transforming in the irrep  $\begin{array}{|c|c|} \hline \square & \square \\ \hline \square & \\ \hline \end{array}$ .
- d) Give a tentative physical interpretation of your results. We will discuss this topic in more detail in Sec. 8.7.4 of the lecture.