Universität Regensburg, Institut für Theoretische Physik WS 2025/26
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Group theory for physicists
Problem set 8 (for the exercises in the week of Dec. 8)

Problem 1 Properties of the projection operators

In Sec. 4.2 of the lecture we introduced the generalized projection operators

$$P^\mu_{ji} \equiv \frac{n_\mu}{n} \sum_{g\in G} [D^\mu(g)^{-1}]_{ji} U(g)\,.$$

Prove the following properties:

$$\begin{aligned} P^\mu_{ji} P^\nu_{\ell k} &= \delta_{\mu\nu}\delta_{jk} P^\mu_{\ell i}\,, \\ U(g) &= \sum_{\mu,i,j} P^\mu_{ji} D^\mu(g)_{ij}\,, \\ U(g) P^\nu_{\ell k} &= \sum_i P^\nu_{\ell i} D^\nu(g)_{ik}\,. \end{aligned}$$

Problem 2 Reduction of a product representation

Let $G = S_3$ and $V = V_2 \otimes V_2$, where V_2 is the 2-dimensional vector space of Problem 6.2 a). Decompose V into irreducible subspaces by acting with the generalized projection operators on the basis vectors $\hat{x}\otimes\hat{x}$, $\hat{x}\otimes\hat{y}$, $\hat{y}\otimes\hat{x}$, and $\hat{y}\otimes\hat{y}$. Determine the Clebsch-Gordan coefficients for this case.

Problem 3 Irreducible operators

In Sec. 4.3 of the lecture we defined irreducible operators, which transform under a group G as

$$U(g) O^\mu_i U(g)^{-1} = O^\mu_j D^\mu(g)_{ji} \quad \text{for all } g \in G\,,$$

where $D^\mu(G)$ is an irreducible matrix representation. We can write this without indices in the form

$$U(g) O U(g)^{-1} = O D(g)\,. \qquad (*)$$

a) Consider the following 6 alternatives to replace the RHS of $(*)$:

(i) $OD(g)^*$	(ii) $OD(g^{-1})$	(iii) $OD(g)^\dagger$
(iv) $D(g)^T O$	(v) $D(g^{-1})O$	(vi) $D(g)^\dagger O$

(The order of the two factors is important because of the implied matrix multiplication. For the time being, do not assume that D is unitary.) In which of these 6 cases do we get representations of the group G on the space of the linear operators $\{O^\mu_i\}$, i.e., sensible alternatives to the original definition $(*)$?

Hint: Recall what condition a representation needs to satisfy.

b) Consider the original definition $(*)$ and the sensible alternatives from part a). Which of these are equivalent to one another if

   i) $D(G)$ is unitary,
   ii) $D(G)$ is equivalent to $D(G)^*$,
   iii) $D(G)$ is both unitary and equivalent to $D(G)^*$?