Universität Regensburg, Institut für Theoretische Physik
Prof. Dr. Tilo Wettig WS 2025/26

**Group theory for physicists**

**Problem set 3 (for the exercises in the week of Nov. 3)**

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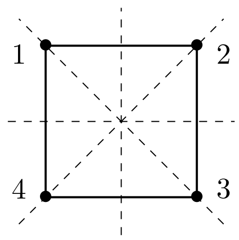
### Problem 1 Factor group

$H_2 = \{I, (123), (321)\}$ is an invariant subgroup of $S_3$. It follows from the discussion in Sec. 1.8 of the lecture that $S_3/H_2$ is a group (the factor group). Use the multiplication law for cosets introduced in Sec. 1.8 to construct the multiplication table of the factor group and show that $S_3/H_2$ is isomorphic to $\mathbb{Z}_2$.

### Problem 2 The group $D_4$

Consider the dihedral group $D_4$, which is the symmetry group of the square, consisting of rotations around the center and reflections about the vertical, horizontal, and diagonal axes.




a) Enumerate the group elements and construct the multiplication table.

b) Find all classes.

c) Find all nontrivial subgroups. Which of them are invariant?

d) For all invariant subgroups $H_i$ from part c), identify the factor group $D_4/H_i$ (i.e., find the group to which $D_4/H_i$ is isomorphic).

e) Is the full group isomorphic to the direct product of some of its subgroups?

### Problem 3 Homomorphism between $\mathrm{Sl}(2, \mathbb{C})$ and the Lorentz group

Let $x = (x_0, x_1, x_2, x_3)$ be a four-vector in Minkowski space $M$. The space $M$ is a 4-dimensional real vector space with metric $||x||^2 = x_0^2 - x_1^2 - x_2^2 - x_3^2$. A homogeneous Lorentz transformation $\Lambda$ is a linear map from $M$ to $M$ that preserves the metric, i.e.,

$$||\Lambda x||^2 = ||x||^2 \qquad \text{for all } x \in M\,.$$

Such transformations form a group, the Lorentz group. Now identify every point in $M$ with a Hermitian $2 \times 2$ matrix,

$$X = \begin{pmatrix} x_0 + x_3 & x_1 - ix_2 \\ x_1 + ix_2 & x_0 - x_3 \end{pmatrix}. \tag{1}$$

It is straightforward to show that $\det X = x_0^2 - x_1^2 - x_2^2 - x_3^2 = ||x||^2$. The set of Hermitian $2 \times 2$ matrices is a 4-dimensional real vector space. A basis of this space is given by the identity $\mathbb{1}_2$

and the three Pauli matrices, i.e., Eq. (1) reads $X = x_0 \mathbb{1}_2 + x_1\sigma^1 + x_2\sigma^2 + x_3\sigma^3$. Now let $A$ be an arbitrary complex $2 \times 2$ matrix and define the action of $A$ on $X$ by

$$X \to AXA^\dagger \,.$$

Since $X$ is related to $x$ via Eq. (1), this induces an action of $A$ on the four-vector $x$,

$$x \to \varphi(A)x \,.$$

Because of $(AXA^\dagger)^\dagger = AXA^\dagger$ the matrix $AXA^\dagger$ is Hermitian, and thus $\varphi(A)x$ is again a real four-vector. Now let us restrict $A$ to be an element of $\mathrm{Sl}(2,\mathbb{C})$, the group of two-dimensional complex matrices with determinant equal to 1. Because of $\det(AXA^\dagger) = |\det A|^2 \det X$ we then have

$$||\varphi(A)x||^2 = ||x||^2 \,,$$

and hence $\varphi(A)$ corresponds to a Lorentz transformation. In fact, the map $\varphi$ is a homomorphism between $\mathrm{Sl}(2,\mathbb{C})$ and the Lorentz group because it preserves the group multiplication:

$$\begin{aligned} &(AB)X(AB)^\dagger = ABXB^\dagger A^\dagger = A(BXB^\dagger)A^\dagger \\ \to \quad &\varphi(AB)x = \varphi(A)\varphi(B)x \,. \end{aligned}$$

a) Show that for the matrix

$$U_\theta = \begin{pmatrix} e^{-i\theta} & 0 \\ 0 & e^{i\theta} \end{pmatrix}$$

   $\varphi(U_\theta)$ is a rotation about the $x_3$-axis by an angle of $2\theta$.

b) Show that for the matrix

$$V_\alpha = \begin{pmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{pmatrix}$$

   $\varphi(V_\alpha)$ is a rotation about the $x_2$-axis by an angle of $2\alpha$.

c) Show that for the matrix

$$M_r = \begin{pmatrix} r & 0 \\ 0 & \frac{1}{r} \end{pmatrix}$$

   $\varphi(M_r)$ is a Lorentz boost in the $x_3$-direction with parameter $2\ln(r)$.

   Hint: A Lorentz boost in the $x_3$-direction with parameter $t$ has the form

$$\begin{pmatrix} \cosh t & 0 & 0 & \sinh t \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ \sinh t & 0 & 0 & \cosh t \end{pmatrix} .$$