Universität Regensburg, Institut für Theoretische Physik — WS 2024/25
Prof. Dr. Tilo Wettig

# Group theory for physicists

## Problem set 13 (for the exercises in the week of Feb. 3)

---

The first 4 problems are regular problems. The remaining 3 problems are optional. You are encouraged to do them once we have covered the corresponding material in the lecture.

### Problem 1 Totally antisymmetric tensors

Show that in the tensor space $V_m^n$ there is no totally antisymmetric tensor of rank $n > m$.

### Problem 2 Irreducibility under Gl(2)

a) Consider the example from Sec. 8.3.3 of the lecture and show that the subspace $T'_\kappa(1) \equiv \{e_\kappa|\alpha\rangle\ ;\ |\alpha\rangle \in V_2^3\}$ is two-dimensional and is spanned by the tensors $|\kappa,1,1\rangle$ and $|\kappa,2,1\rangle$ of symmetry $\Theta_\kappa$. Here, $e_\kappa$ is the Young operator corresponding to the normal Young tableau $\Theta_\kappa = \begin{array}{|c|c|}\hline 1 & 2 \\ \hline 3 \\ \cline{1-1}\end{array}$.

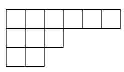
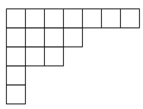
b) Show that the subspace $T'_\kappa(1)$ is irreducible under Gl(2).

### Problem 3 Equivalence of irreps of SU($m$)

a) Show that for Gl($m$) the Young diagram with 1 column and $m$ rows corresponds to the 1-dimensional irrep of $g \in \mathrm{Gl}(m)$ given by $g \to \det(g)$.

b) Show that for SU($m$) this is the trivial representation.

c) Consider irreps of Gl($m$) given by the Young diagrams $\Theta = (\lambda_1, \ldots, \lambda_m)$ and $\Theta' = (\lambda_1 + 1, \ldots, \lambda_m + 1)$. E.g., for $n = 11$, $m = 5$, and $\Theta = (6, 3, 2, 0, 0)$ these are the diagrams

$\Theta =$  and $\Theta' =$ 

Show that $\Theta' =$  $\otimes\, \Theta$, where the first diagram on the RHS has $m$ rows.

d) Show that for SU($m$) $\Theta'$ and $\Theta$ are equivalent.

e) Show that for SU($m$) $\Theta'$ and $\Theta$ are also equivalent if $\Theta'$ contains $k$ additional columns with $m$ boxes each.

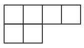
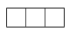
### Problem 4 Pseudoreality of SU(2)

Show that for SU(2) the irreps 2 and $\bar{2}$ are equivalent.
Hint: Consider first the matrix

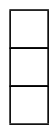
$$C = i\sigma_2 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

and show that $C\vec{\sigma}C^{-1} = -\vec{\sigma}^*$. Then show that for an arbitrary element $U = \exp(-\frac{i}{2}\vec{\sigma}\cdot\vec{\alpha})$ of SU(2) we have $CUC^{-1} = U^*$.

## Problem 5 Reduction of product representations of SU($m$)

a) Decompose the product representation  $\otimes$  of SU(3) into irreps of SU(3) and compute the dimensions of all irreps that appear in the decomposition.
(Result: $27 \otimes 10 = 81 \oplus 64 \oplus 35 \oplus \overline{35} \oplus 27 \oplus 10 \oplus \overline{10} \oplus 8$.)

b) Same as a), but now for SU(4) and  $\otimes$ .

c) Same as a), but now for SU(5) and  $\otimes$ .

## Problem 6 Baryon decuplet and octet

Consider SU(3)$_{\text{flavor}}$ and the 27 product states $uuu$, $uud$, $\ldots$, $sss \in V_3^3$. Construct the basis tensors of the irreducible subspaces under SU(3) in the decomposition

$$3 \otimes 3 \otimes 3 = 10 \oplus 8 \oplus 8 \oplus 1$$

using the methods from Sec. 8.3 of the lecture. Compare your solution with problem 10.4 and discuss the connection between the two problems.

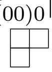
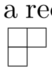
## Problem 7 The Gell-Mann–Okubo formula

In a baryon multiplet the masses of all particles in the same isospin multiplet are roughly equal, while there are larger mass differences between different isospin multiplets. This is because the strange quark is significantly heavier than the up and down quarks so that the SU(3)$_{\text{flavor}}$ symmetry is explicitly broken. Since isospin and hypercharge are conserved by the strong interactions, the remaining symmetry group is SU(2)$_{\text{isospin}} \otimes$ U(1)$_{\text{hypercharge}}$.
We assume that the Hamiltonian has the form $H = H_0 + H'$, where $H_0$ is invariant under SU(3)$_{\text{flavor}}$ and $H'$ is a small perturbation. Every baryon is described by a normalized state $|\psi\rangle$, with mass $m = \langle\psi|H|\psi\rangle$.
The unperturbed states, which we denote by $\psi_i^\lambda \equiv \psi_{(I_3 Y)I}^\lambda$, form SU(3) multiplets, where $\lambda$ labels the irrep of SU(3). If we only consider $H_0$, the unperturbed states have the degenerate mass $\langle\psi_i^\lambda|H_0|\psi_i^\lambda\rangle = a_\lambda$. The mass differences within a multiplet are given in first-order degenerate perturbation theory by the eigenvalues of the matrix $M_{ij}^\lambda = \langle\psi_i^\lambda|H'|\psi_j^\lambda\rangle$. Because of isospin and hypercharge conservation this matrix is diagonal, i.e.,

$$\Delta m_i^\lambda = \langle\psi_i^\lambda|H'|\psi_i^\lambda\rangle \,.$$

a) $H'$ can be expanded in irreducible operators under SU(3). Show that the singlet contribution of this expansion does not yield mass differences within a multiplet, but only a constant shift that can be absorbed in $a_\lambda$.

b) Conclude from the invariance of $H'$ under SU(2)$_I \otimes$U(1)$_Y$ that $H'$ can only contain operators $O_k^\mu$ corresponding to the state with $Y = I = I_3 = 0$, i.e., $k = (00)0$. Show that for this reason the triplet (dim = 3), sextet (dim = 6), and decuplet (dim = 10) representations do not give a contribution to $H'$. If we neglect higher-dimensional representations, $H'$ thus only contains operators that transform like $O_{(00)0}^8$.

c) In the formula $\Delta m_i^\lambda = \langle\psi_i^\lambda|O_{(00)0}^8|\psi_i^\lambda\rangle$ we first consider the object $O_{(00)0}^8|\psi_i^\lambda\rangle$. It transforms in the product representation  $\otimes D^\lambda$ (see Sec. 4.3 of the lecture). If we decompose the latter into irreducible components we only need the component $D^\lambda$, since we subsequently take the scalar product with $\langle\psi_i^\lambda|$.
For now consider only irreps $D^\lambda$ corresponding to a rectangular Young diagram with at most two rows and show that in this case the product  $\otimes D^\lambda$ contains the irrep $D^\lambda$ once and

only once. Use the graphical rule from Sec. 8.6 of the lecture. Consider separately diagrams with one row and with two rows.

d) For our problem the Wigner-Eckart theorem (see Sec. 4.3 of the lecture) has the form

$$\langle\psi_i^\lambda|O_k^8|\psi_i^\lambda\rangle = \sum_\alpha \langle\alpha,\lambda,i(8,\lambda)k,i\rangle\langle\lambda||O^8||\lambda\rangle_\alpha$$

with $k=(00)0$. The result of part c) implies that on the RHS the summation index $\alpha$ only takes on one value (provided that $\lambda$ corresponds to a rectangular Young diagram). The Wigner-Eckart theorem thus shows that every operator transforming like $O^8_{(00)0}$ yields the same $Y$-dependence of $\Delta m_i^\lambda$ (why?).
The 8 generators of SU(3) are given by $S_i = \frac{1}{2}\lambda_i$ with the Gell-Mann matrices

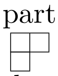
$$\lambda_i = \begin{pmatrix} \multicolumn{2}{c}{\sigma_i} & 0 \\ & & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad \text{for } i=1,2,3, \quad \lambda_4 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \quad \lambda_5 = \begin{pmatrix} 0 & 0 & -i \\ 0 & 0 & 0 \\ i & 0 & 0 \end{pmatrix},$$

$$\lambda_6 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}, \quad \lambda_7 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -i \\ 0 & i & 0 \end{pmatrix}, \quad \lambda_8 = \frac{\sqrt{3}}{3}\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{pmatrix}.$$

Here, the $S_i$ for $i=1,2,3$ are the isospin generators, and $\hat{Y} = \frac{2}{\sqrt{3}}S_8$ is the hypercharge generator. The 8 generators transform in the adjoint representation, which for SU(3) corresponds to the octet representation. Show that $S_8$ commutes with the isospin generators and therefore corresponds, in analogy to part b), to the octet state with $Y=I=I_3=0$. Thus $S_8 = \frac{\sqrt{3}}{2}\hat{Y}$ is a permissible $O^8_{(00)0}$ operator, so that $\Delta m_i^\lambda \propto \langle\psi_i^\lambda|S_8|\psi_i^\lambda\rangle$ and hence

$$\Delta m_i^\lambda = b_\lambda\, Y \quad \text{for rectangular Young diagrams } \Theta_\lambda$$

with a factor $b_\lambda$ that is constant for a given multiplet.

e) Now consider, in analogy to part c), Young diagrams that are not rectangular and show that in this case the product  $\otimes D^\lambda$ contains the irrep $D^\lambda$ exactly twice. Also discuss what happens if $\lambda$ is the trivial representation of SU(3).

f) It follows from the result of part e) and from the Wigner-Eckart theorem that we now need two operators that transform like $O^8_{(00)0}$ to obtain a sufficiently general form of $H'$ (why are two operators sufficient?). In the following we construct two sets of operators that transform as octets. Because of $[S_i, S_j] = \sum_k c_{ij}^k S_k$ (see Sec. 6.5 of the lecture) one of these sets is identical with the generators, i.e., $S_8$ (or $\hat{Y}$) is one of the desired operators. The second set is given by the operators

$$D_i = \frac{2}{3}\sum_{jk} d_{ijk} S_j S_k$$

with constants $d_{ijk}$ given by

$$S_i S_j + S_j S_i = \sum_k d_{ijk} S_k + \frac{1}{3}\delta_{ij}\mathbb{1}\,.$$

We don't need to know the $d_{ijk}$ explicitly. Show that the $D_i$ indeed transform like an octet, i.e.,

$$[S_i, D_j] = \sum_k c_{ij}^k D_k\,.$$

Thus the second desired operator is $D_8$. We could now compute $D_8$ explicitly but choose another method in the following.

g) Show that $D_8$ commutes with $S_8$ and the isospin generators and therefore can only depend on the operators $\hat{I}^2$, $\hat{Y}^2$, and the quadratic Casimir operator $C_2 = \sum_{i=1}^{8} S_i^2$. Thus the $D_8$ contributon to $\Delta m_i^\lambda$ has the form $c_\lambda\big[I(I+1) + dY^2 + \text{const}\big]$, where the constant can be absorbed in $a_\lambda$. Why does $\hat{I}_3^2$ not occur?

h) To compute $d$ we again consider rectangular Young diagrams for which part d) implies that $I(I+1) + dY^2$ must be linear in $Y$. Show for Young diagrams with one row that for the isospin multiplets we have $I = Y/2 + \text{const.}$, and for rectangular Young diagrams with two rows that $I = -Y/2 + \text{const.}$ To do so, use the corresponding weight diagrams. (Hint: In the first case the weight diagrams are triangles with the tip down, in the second case triangles with the tip up.) In both cases the quantity $I(I+1) - Y^2/4$ is linear in $Y$. This finally yields the Gell-Mann–Okubo formula

$$m_i^\lambda = a_\lambda + b_\lambda Y + c_\lambda \left[ I(I+1) - \frac{Y^2}{4} \right] .$$

This formula predicts a mass of 1680 MeV for the $\Omega^-$ particle, which had not been found in experiments at the time the prediction was made. Subsequently the $\Omega^-$ was found, with mass 1672 MeV — a great success of group theory.