Universität Regensburg, Institut für Theoretische Physik
Prof. Dr. Tilo Wettig WS 2024/25

Group theory for physicists

Problem set 11 (for the exercises in the week of Jan. 20)

Remark: Because of the holiday on Jan. 6 we are one exercise short. Therefore we have to skip two problems that were announced in class: the relation between Euler angles and angle-and-axis parameters for SO(3), and the proof that the generator of a rotation about the axis $\hat{n} = n_k \hat{e}_k$ is $J_{\hat{n}} = n_k J_k$. Both are discussed in Tung's book, see Eq. (7.1-14) and Theorem 7.2.

Problem 1 Rotations by the same angle are in the same class

Show that for any $R \in \mathrm{SO}(3)$ we have $R R_{\hat{n}}(\psi) R^{-1} = R_{\hat{n}'}(\psi)$ with $\hat{n}' = R\hat{n}$.

Problem 2 SU(2) matrices as exponentials of the generators

Show that for $\vec{\alpha} = \hat{n}\psi$ we have

$$e^{-i\frac{1}{2}\vec{\sigma}\cdot\vec{\alpha}} = \mathbb{1}\cos\frac{\psi}{2} - i\vec{\sigma}\cdot\hat{n}\sin\frac{\psi}{2}\,.$$

Here, $\hat{n}$ is a unit vector in three dimensions, and the vector $\vec{\sigma}$ contains the three Pauli matrices

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \qquad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

Hint: You can solve this problem without knowing anything about SU(2). First show $(\vec{\sigma}\cdot\hat{n})^2 = \mathbb{1}$.

Problem 3 Invariant integration measure for SU(2)

Compute the Haar measure for the integration over SU(2) in the standard parameterization of SU(2) from Sec. 6.8.1 of the lecture,

$$A = \begin{pmatrix} \cos\theta\, e^{i\zeta} & -\sin\theta\, e^{i\eta} \\ \sin\theta\, e^{-i\eta} & \cos\theta\, e^{-i\zeta} \end{pmatrix}.$$

Normalize the integration measure such that $\mathrm{vol}(\mathrm{SU}(2)) = 1$.

Hints: The generators of SU(2) are the three Pauli matrices σ_1, σ_2 and σ_3. Use the general method to construct the Haar measure from Sec. 6.8.3 of the lecture.

Problem 4 SO(3) and SU(2)

a) Construct the irrep D^j of SO(3) for $j = \frac{1}{2}$. Start with the general expression

$$D^j(\alpha,\beta,\gamma)_{m'm} = e^{-i(\alpha m' + \gamma m)}\langle jm'|e^{-i\beta J_2}|jm\rangle$$

from Sec. 6.7.4 of the lecture.

Hint: Using the action of $J_\pm$ on the states $|\frac{1}{2}, \pm\frac{1}{2}\rangle$, construct the explicit form of J_2 in this two-dimensional subspace.

b) Convert the standard parameterization of SU(2) from Sec. 6.8.1 of the lecture (see problem 3) to the form $D^{1/2}(\alpha,\beta,\gamma)$. What are the intervals of the three angles? Compare with the intervals of the Euler angles for SO(3) and discuss the difference.