Universität Regensburg, Institut für Theoretische Physik
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Group theory for physicists
Problem set 9 (for the exercises in the week of Dec. 16)

Problem 1 Reduction of the regular representation of C_3

Consider the Abelian group C_3 with elements e, a, a^{-1} (e is the identity).

a) How many irreps does C_3 have, what are their dimensions, and how often do they occur in the regular representation?

b) Show that

$$e_1 = \frac{1}{3}(e + a + a^{-1})$$

is a primitive idempotent and generates the trivial representation.

c) Now consider another idempotent

$$e_2 = xe + ya + za^{-1}$$

and use the conditions $e_1 e_2 = 0$ and $e_2 e_2 = e_2$ to derive equations for x, y, and z. Determine all solutions of these equations and find out whether the resulting idempotents are primitive.

d) Find out whether the primitive idempotents generate equivalent or inequivalent representations.

e) Construct all irreducible left ideals and the corresponding irreps of C_3. Summarize the latter in a table.

Problem 2 Product representations of S_n

Prove the following theorem: If D^λ and D^μ are two irreps of S_n, then

a) $D^\lambda \otimes D^\mu$ contains D^s with multiplicity one (zero) if D^λ and D^μ are equivalent (inequivalent).

b) $D^\lambda \otimes D^\mu$ contains D^a with multiplicity one (zero) if D^λ and D^μ are associated (not associated).

Hint: For the multiplicity, use $a_k = \frac{1}{n!}\sum_p \chi^k(p)^* \chi(p)$. Also, use $\chi^{\lambda\otimes\mu} = \chi^\lambda \chi^\mu$ and the fact that for S_n the characters are real.

Problem 3 Symmetrizer and antisymmetrizer of S_n

Show that the symmetrizer $s = \sum\limits_p p$ and the antisymmetrizer $a = \sum\limits_p (-1)^p p$ of S_n are essentially idempotent and primitive.

Problem 4 Irreducible symmetrizers of S_3

The irreducible symmetrizers of the standard Young tableaux of S_3 are given by

$$
\begin{aligned}
e_1 &= s\,, \qquad & e_2 &= e + (12) - (13) - (321)\,,\\
e_3 &= a\,, \qquad & e_2^{(23)} &= e - (12) + (13) - (123)\,,
\end{aligned}
$$

see Sec. 5.4 of the lecture. Show that these four elements of the group algebra $\tilde{S}_3$ are essentially idempotent and primitive.