Universität Regensburg, Institut für Theoretische Physik WS 2024/25
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Group theory for physicists

Problem set 8 (for the exercises in the week of Dec. 9)

Problem 1 Properties of the projection operators

Prove the following properties of the generalized projection operators introduced in Sec. 4.2 of the lecture:

$$P^\mu_{ji} P^\nu_{\ell k} = \delta_{\mu\nu} \delta_{jk} P^\mu_{\ell i} \,,$$
$$U(g) = \sum_{\mu,i,j} P^\mu_{ji} D^\mu(g)_{ij} \,,$$
$$U(g) P^\nu_{\ell k} = \sum_i P^\nu_{\ell i} D^\nu(g)_{ik} \,.$$

Problem 2 Reduction of a product representation

Let $G = S_3$ and $V = V_2 \otimes V_2$, where V_2 is the 2-dimensional vector space of Problem 6.2 a). Decompose V into irreducible subspaces by acting with the generalized projection operators on the basis vectors $\hat{x} \otimes \hat{x}$, $\hat{x} \otimes \hat{y}$, $\hat{y} \otimes \hat{x}$, and $\hat{y} \otimes \hat{y}$. Determine the Clebsch-Gordan coefficients for this case.

Problem 3 Irreducible operators

In Sec. 4.3 of the lecture we defined irreducible operators, which transform under a group G as

$$U(g) O^\mu_i U(g)^{-1} = O^\mu_j D^\mu(g)_{ji} \quad \text{for all } g \in G \,,$$

where $D^\mu(G)$ is an irreducible matrix representation. We can write this without indices in the form

$$U(g) O U(g)^{-1} = O D(g) \,. \qquad (*)$$

a) Consider the following 6 alternatives to replace the RHS of $(*)$:

(i) $OD(g)^*$	(ii) $OD(g^{-1})$	(iii) $OD(g)^\dagger$
(iv) $D(g)^T O$	(v) $D(g^{-1})O$	(vi) $D(g)^\dagger O$

(The order of the two factors is important because of the implied matrix multiplication. For the time being, do not assume that D is unitary.) In which of these 6 cases do we get representations of the group G on the space of the linear operators $\{O^\mu_i\}$, i.e., sensible alternatives to the original definition $(*)$?

Hint: Recall what condition a representation needs to fulfill.

b) Consider the original definition $(*)$ and the sensible alternatives from part a). Which of these are equivalent to each other if

i) $D(G)$ is unitary,
ii) $D(G)$ is equivalent to $D(G)^*$,
iii) $D(G)$ is both unitary and equivalent to $D(G)^*$?