Universität Regensburg, Institut für Theoretische Physik WS 2024/25
Prof. Dr. Tilo Wettig

Group theory for physicists
Problem set 7 (for the exercises in the week of Dec. 2)

General remark: Whenever possible, please try to check your results by computer, using numerical methods or a computer algebra system. This is beneficial in at least three respects: (1) you can verify your results, (2) you typically learn more about the subject, and (3) you gain experience with programming. There are many suitable high-level languages such as python/sympy, matlab/octave, maple, mathematica, GAP, and others.

Problem 1 The regular representation of S_3

a) Construct the regular representation of S_3, i.e., the six 6×6 matrices corresponding to the six group elements.

b) Show that the regular representation contains the three irreps of S_3 with the expected multiplicities (= dimensions of the irreps).

Problem 2 Characters of product representations

Show that the characters of the product representation $\Gamma^{\mu\otimes\nu} = \Gamma^\mu \otimes \Gamma^\nu$ of a group G satisfy the relation

$$\chi^{\mu\otimes\nu}(g) = \chi^\mu(g)\chi^\nu(g)$$

for all $g \in G$.

Problem 3 Product wave functions for spin

In quantum mechanics one defines the spin wave function of the electron by $|\alpha\rangle$ = "spin down" and $|\beta\rangle$ = "spin up" ($|\alpha\rangle$ and $|\beta\rangle$ are orthonormal). For a system of three electrons we can define product wave functions of the form $|\alpha(1)\alpha(2)\alpha(3)\rangle$, $|\alpha(1)\alpha(2)\beta(3)\rangle$, etc. (there are $2^3 = 8$ possibilities). These 8 functions are orthonormal to one another and span a space that is invariant under S_3. Therefore they furnish an 8-dimensional representation of S_3.

a) Construct this 8-dimensional representation, i.e., consider the action of the group elements on the 8 functions and read off the representation matrices.

b) Find out which of the irreps of S_3 are contained in this 8-dimensional representation (including the corresponding multiplicities).

c) Identify the carrier spaces of the trivial representation.

Remark: In quantum mechanics an odd number of fermion interchanges gives a minus sign. Here, however, we just consider the action of S_3 without minus signs.

Problem 4 Induced representations

a) Every group G has as a subgroup a one-element group consisting of the identity element alone. This one-element group has a single irrep in which the identity element is represented by 1. Show that the representation of the full group G induced by this irrep of the one-element subgroup is the regular representation.

b) Use the Frobenius reciprocity theorem to determine the multiplicities with which the irreps of G occur in the induced representation from a).