Universität Regensburg, Institut für Theoretische Physik
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Group theory for physicists

Problem set 6 (for the exercises in the week of Nov. 25)

Problem 1 Number of irreps = number of classes

Let G be a group of order n. Furthermore, let m be the number of classes and p be the number of non-equivalent irreps of G. In Sec. 2.4.4 of the lecture we have only shown $p \leq m$. Here we prove the equal sign.

a) The orthogonality relations for the matrix elements of the irreps are (see Sec. 2.4.3)

$$\frac{\lambda_i}{n}\sum_g \Gamma^i(g)^*_{\mu\nu}\Gamma^k(g)_{\mu'\nu'} = \delta_{ik}\delta_{\mu\mu'}\delta_{\nu\nu'} \qquad (*)$$

with $\lambda_i = \dim(\Gamma^i)$. As in the lecture we now collect (for fixed i,μ,ν) the n numbers $\Gamma^i(g_1)_{\mu\nu}$ to $\Gamma^i(g_n)_{\mu\nu}$ in a vector with n components. For every irrep Γ^i there are λ_i^2 such vectors. According to $(*)$ all these vectors are orthogonal to one another. In Sec. 2.5 we have shown

$$\sum_{i=1}^{p} \lambda_i^2 = n\,.$$

This means that there are n such orthogonal vectors, i.e., n triples (i,μ,ν) which we label by the index $a = 1,\ldots,n$. Furthermore, we define n normalized vectors x^a with components

$$x^a_j = \sqrt{\frac{\lambda_i}{n}}\,\Gamma^i(g_j)_{\mu\nu} \qquad (a, j = 1,\ldots,n)\,.$$

With these definitions, $(*)$ simply means

$$\langle x^a | x^b\rangle = \delta_{ab}\,,$$

where b corresponds to the triple (k,μ',ν'). Using these preliminary results, show that the matrix elements of the irreps satisfy the completeness relation

$$\sum_{i=1}^{p}\sum_{\mu,\nu=1}^{\lambda_i}\frac{\lambda_i}{n}\Gamma^i(g_j)_{\mu\nu}\Gamma^i(g_{j'})^*_{\mu\nu} = \delta_{jj'}\,.$$

Hint: You can show this in one or two lines.

b) For an irrep $\Gamma^i(G)$, show that the sum of the $\Gamma^i(g)$ over the elements of a class c is

$$\sum_{g\in c}\Gamma^i(g) = \frac{n_c}{\lambda_i}\chi^i_c \mathbb{1}_{\lambda_i}\,.$$

Here, n_c is the number of group elements in class c, χ^i_c is the character of class c in irrep Γ^i, and $\mathbb{1}_{\lambda_i}$ is the unit matrix of dimension λ_i.

Hint: Show that the LHS commutes with $\Gamma^i(g)$ for all $g \in G$, use Schur's Lemma 1, and compute the trace of both sides of the equation.

c) From Sec. 2.4.4 we know the orthogonality relations for characters,

$$\sum_{c=1}^{m} \frac{n_c}{n} (\chi_c^i)^* \chi_c^k = \delta_{ik} \,.$$

Show that the characters also satisfy the completeness relation

$$\frac{n_c}{n} \sum_{i=1}^{p} \chi_c^i (\chi_{c'}^i)^* = \delta_{cc'} \,.$$

Hint: In the result of a), sum over g and g' in classes c and c' and use b).

d) Let M be an $m \times p$ matrix with

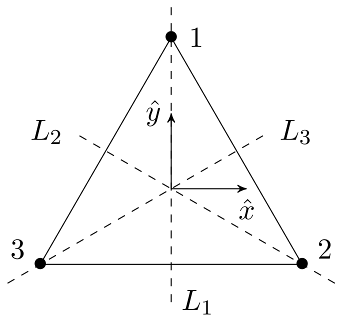
$$M^\dagger M = \mathbb{1}_p \quad \text{and} \quad MM^\dagger = \mathbb{1}_m \,.$$

Show that this is only possible for $m = p$.

e) Use c) and d) to show that the number of non-equivalent irreps of G equals the number of classes.

Problem 2 A product representation of D_3

a) Construct the matrix representation of D_3 in the 2-dimensional space spanned by the unit vectors $\hat{x}$ and $\hat{y}$ (see figure). Is this representation irreducible?



b) Let (x_1, y_1) and (x_2, y_2) be the coordinates of two vectors that transform under D_3 independently and as in part a). Consider the 4-dimensional space V spanned by the functions $x_1 x_2$, $x_1 y_2$, $y_1 x_2$, and $y_1 y_2$. Construct the representation of D_3 on V and show that it is the product of the representation of part a) with itself.

c) Find the irreps of D_3 contained in the 4-dimensional product representation of part b).

Problem 3 Clebsch-Gordan coefficients

Prove the following orthonormality and completeness relations for the Clebsch-Gordan coefficients introduced in Sec. 2.6 of the lecture:

$$\sum_{\alpha\lambda\ell} \langle i', j'(\mu,\nu)\alpha, \lambda, \ell\rangle \langle \alpha, \lambda, \ell(\mu,\nu) i, j\rangle = \delta_{i'i}\delta_{j'j} \,,$$

$$\sum_{ij} \langle \alpha', \lambda', \ell'(\mu,\nu) i, j\rangle \langle i, j(\mu,\nu)\alpha, \lambda, \ell\rangle = \delta_{\alpha'\alpha}\delta_{\lambda'\lambda}\delta_{\ell'\ell} \,.$$

Hint: Use the orthonormality and completeness of the basis systems $\{|i,j\rangle\}$ and $\{|\alpha,\lambda,\ell\rangle\}$.