Universität Regensburg, Institut für Theoretische Physik WS 2024/25
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# Group theory for physicists

## Problem set 5 (for the exercises in the week of Nov. 18)

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### Problem 1 Irreducible representations of Abelian groups

Show that all irreducible representations of Abelian groups have dimension 1.
Hint: Use Schur's Lemma 1.

### Problem 2 $D_4$ again

a) How many non-equivalent irreps does $D_4$ have? What are their dimensions?

b) Construct all one-dimensional irreps.
Hint: Use Problems 3.2 and 4.3a).

c) Construct the character table and the remaining unitary irrep of $D_4$.

### Problem 3 Representations of $D_3$

Consider the 6-dimensional function space $V$ consisting of polynomials of degree 2 in two real variables $x$ and $y$,

$$f(x, y) = ax^2 + bxy + cy^2 + dx + ey + h\,,$$

where $a, \ldots, h$ are complex constants.

a) If $(x, y)$ transforms under $D_3$ as a 2-dimensional vector we obtain a 6-dimensional representation of $D_3$ (see Sec. 2.4.2 of the lecture). Find this 6-dimensional representation.

b) Identify the various irreps of $D_3$ contained in the 6-dimensional representation of part a) and the corresponding invariant subspaces of $V$.

### Problem 4 Proofs of the four theorems from Sec. 2.4.3

These are not assigned as homework but will be presented by the TA and discussed in the exercises. However, you are encouraged to think about them beforehand. If you can present one of the proofs you will get extra credit.

a) Every representation of a group $G$ by matrices with nonzero determinant can be converted by a basis transformation to a representation by unitary matrices.

b) Schur's lemma 1: A matrix which commutes with all matrices of an irrep is proportional to the identity matrix.

c) Schur's lemma 2: Suppose we have two irreps of a group $G$: $\Gamma^1(G)$ with dimension $\lambda_1$ and $\Gamma^2(G)$ with dimension $\lambda_2$. If there exists a $\lambda_2 \times \lambda_1$ matrix $M$ such that

$$M\Gamma^1(g) = \Gamma^2(g)M \qquad \text{for all } g \in G$$

then if $\lambda_1 \neq \lambda_2$ we have $M = 0$, and if $\lambda_1 = \lambda_2$ we have either $M = 0$ or $\det M \neq 0$. If $\det M \neq 0$ the two irreps $\Gamma^1(G)$ and $\Gamma^2(G)$ are equivalent (since $\Gamma^1(G) = M^{-1}\Gamma^2(G)M$).

d) Let $\Gamma^i(G)$ and $\Gamma^k(G)$ be two non-equivalent, unitary irreps of $G$ with $\text{order}(G) = n$. Then

$$\sum_g (\Gamma^i(g)_{\mu\nu})^* \Gamma^k(g)_{\mu'\nu'} = 0 \qquad \text{for all } \mu, \nu, \mu', \nu' \,.$$

For the matrix elements of a single unitary irrep with dimension $\lambda_i$ we have

$$\sum_g (\Gamma^i(g)_{\mu\nu})^* \Gamma^i(g)_{\mu'\nu'} = \frac{n}{\lambda_i} \delta_{\mu\mu'} \delta_{\nu\nu'} \qquad \text{for } \mu, \nu, \mu', \nu' = 1, \ldots, \lambda_i \,.$$