Universität Regensburg, Institut für Theoretische Physik
Prof. Dr. Tilo Wettig WS 2024/25

Group theory for physicists
Problem set 4 (for the exercises in the week of Nov. 11)

Problem 1 Homomorphism and Abelian groups

Let G be a group and define a map $f : G \to G$ by $f(g) = g^2$ for every $g \in G$. Prove that G is an Abelian group if and only if the map f is a group homomorphism.

Problem 2 More group homomorphisms

Let G, H and K be groups. Furthermore, let $f : G \to K$ be a group homomorphism and let $\pi : G \to H$ be a surjective group homomorphism such that the kernel of π is included in the kernel of f, i.e., we have $\ker(\pi) \subseteq \ker(f)$. We can define a map $\bar{f} : H \to K$ as follows. For each $h \in H$, there exists a $g \in G$ such that $\pi(g) = h$ since $\pi : G \to H$ is surjective. The map $\bar{f} : H \to K$ is then defined by $\bar{f}(h) = f(g)$.

a) Prove that the map $\bar{f} : H \to K$ is well-defined. I.e., if two elements $g_1, g_2 \in G$ satisfy $\pi(g_1) = \pi(g_2) = h$, show that $f(g_1) = f(g_2)$.

b) Prove that $\bar{f} : H \to K$ is a group homomorphism.

Problem 3 An important theorem

A representation is called faithful if the homomorphism between the group elements and the representation matrices is one-to-one, i.e., different group elements are represented by different matrices. Prove the following theorem:

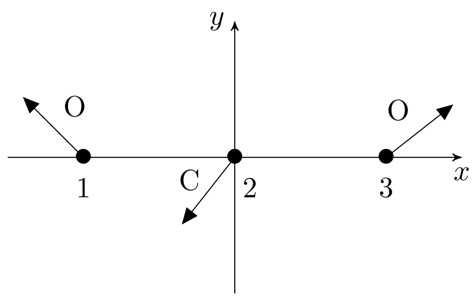
a) If a group G has a nontrivial invariant subgroup H, then a representation of the factor group G/H is also a representation of G. This representation is not faithful.

b) Converse: If $\Gamma(G)$ is an unfaithful representation of G, then G has at least one invariant subgroup H such that Γ defines a faithful representation of the factor group G/H.

   Hint: Use the theorem from Sec. 1.10 of the lecture.

Problem 4 Coplanar vibrations of the CO_2 molecule

CO_2 is a linear molecule. In the ground state the carbon atom is in the middle between the two oxygen atoms. In the following we consider coplanar vibrations of the three atoms in the xy-plane.



The translations of the three atoms can be collected in a 6-dimensional vector $(x_1, y_1, x_2, y_2, x_3, y_3)$. Let the basis vectors of the corresponding space be

$$e_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} \qquad \cdots \qquad e_6 = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} .$$

The symmetry group of the system (the so-called Vierergruppe or Klein group) has 4 elements: the identity (I), reflections about the x- and y-axis (σ_x and σ_y), and a rotation by $180°$ around the origin (R).

a) Determine the action of the 4 group elements on the 6 basis vectors $e_1, \ldots, e_6$.

b) Use this action to derive a 6-dimensional matrix representation of the symmetry group.

c) Determine the multiplication table of the symmetry group from a) or b).

d) Is the representation in part b) reducible?
Hint: For this part you may have to wait for Sec. 2.4.3 of the lecture.