Universität Regensburg, Institut für Theoretische Physik
Prof. Dr. Tilo Wettig WS 2024/25

Group theory for physicists

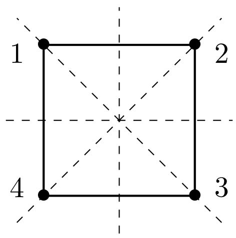
Problem set 3 (for the exercises in the week of Nov. 4)

Problem 1 Factor group

$H_2 = \{I, (123), (321)\}$ is an invariant subgroup of S_3. It follows from the discussion in Sec. 1.8 of the lecture that S_3/H_2 is a group (the factor group). Use the multiplication law for cosets introduced in Sec. 1.8 to construct the multiplication table of the factor group and show that S_3/H_2 is isomorphic to $\mathbb{Z}_2$.

Problem 2 The group D_4

Consider the dihedral group D_4, which is the symmetry group of the square, consisting of rotations around the center and reflections about the vertical, horizontal, and diagonal axes.



a) Enumerate the group elements and construct the multiplication table.

b) Find all classes.

c) Find all nontrivial subgroups. Which of them are invariant?

d) For all invariant subgroups H_i from part c), identify the factor group D_4/H_i (i.e., find the group to which D_4/H_i is isomorphic).

e) Is the full group isomorphic to the direct product of some of its subgroups?

Problem 3 Homomorphism between Sl(2, $\mathbb{C}$) and the Lorentz group

Consider the homomorphism between Sl(2, $\mathbb{C}$) and the Lorentz group introduced in Sec. 1.11 of the lecture, which identifies every four-vector $x = (x_0, x_1, x_2, x_3)$ with a Hermitian matrix

$$X = \begin{pmatrix} x_0 + x_3 & x_1 - ix_2 \\ x_1 + ix_2 & x_0 - x_3 \end{pmatrix}.$$

The action $X \to AXA^\dagger$ of $A \in \mathrm{Sl}(2, \mathbb{C})$ on X induces a Lorentz transformation $x \to \varphi(A)x$ of the four-vector x.

a) Show that for the matrix

$$U_\theta = \begin{pmatrix} e^{-i\theta} & 0 \\ 0 & e^{i\theta} \end{pmatrix}$$

$\varphi(U_\theta)$ is a rotation about the x_3-axis by an angle of 2θ.

b) Show that for the matrix

$$V_\alpha = \begin{pmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{pmatrix}$$

$\varphi(V_\alpha)$ is a rotation about the x_2-axis by an angle of 2α.

c) Show that for the matrix

$$M_r = \begin{pmatrix} r & 0 \\ 0 & \frac{1}{r} \end{pmatrix}$$

$\varphi(M_r)$ is a Lorentz boost in the x_3-direction with parameter $2\ln(r)$.

Hint: A Lorentz boost in the x_3-direction with parameter t has the form

$$\begin{pmatrix} \cosh t & 0 & 0 & \sinh t \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ \sinh t & 0 & 0 & \cosh t \end{pmatrix}.$$