Übungen zur Quantenmechanik für LA und Nanoscience
Blatt 6 (für die Übungen in der Woche 27.11.-01.12.)

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1 Continuity equation in an absorptive potential

Consider the generalization of the continuity equation to the case of a complex potential $V(x) = V_r(x) - iV_i$, where V_r and V_i are real, and V_i is assumed to be constant and positive.

a) What does this modification imply for the properties of the Hamilton operator?

b) Generalize the derivation of the continuity equation, see Sec. 3.7 of the lecture, for this case. (The definition of $\vec{J}$ remains unchanged.)

c) Show that the total probability for finding the particle decreases as $e^{-2V_i t/\hbar}$.

d) Give a physical interpretation of the result of part c).

2 Ground-state energy of the hydrogen atom

The uncertainty relation is frequently used to obtain semiquantitative estimates of quantum-mechanical effects. One of these is the stability of the hydrogen atom. The latter is classically unstable since the electron, moving along a closed trajectory in the attractive Coulomb potential $-e^2/r$ of the proton, would lose energy by the emission of electromagnetic radiation and eventually fall into the nucleus. Simple arguments based on the uncertainty principle hint at the existence of a stable ground state.

a) Assume that the electron is described by a spherically symmetric wave function whose spatial extent is characterized by a radius a (i.e., the probability of finding the electron at a distance larger than a from the nucleus is so small that a corresponds to the size of the atom). Use the uncertainty relation to determine the energy

$$E = \langle T \rangle + \langle V \rangle$$

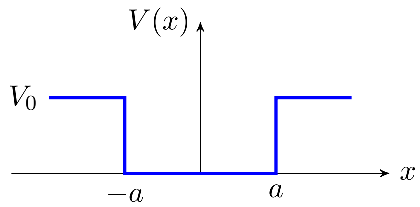
of the electron, where $\langle T \rangle$ and $\langle V \rangle$ are the expectation values of the kinetic and potential energy, respectively. Draw a sketch of $\langle T \rangle$, $\langle V \rangle$ and E as a function of a.

b) Determine the radius a_0 for which the energy assumes a minimum and compute the corresponding energy E_0 as an estimate of the true ground-state energy. Compare your estimate with the exact result of $-13.6\,\mathrm{eV}$.

3 Potential well

Consider a particle of mass m in a square-well potential

$$V(x) = \begin{cases} 0\,, & |x| < a\,, \\ V_0 > 0\,, & |x| \geq a\,. \end{cases}$$



Unlike in the case of the box potential (where $V_0 \to \infty$), one finds both bound states and unbound states (also called scattering states). In the following we only consider the bound states, which necessarily have energies $E \leq V_0$. The lowest states should be very similar to those of the box potential. The main difference will be that they do not vanish at the boundaries but have (exponentially decaying) tails that reach beyond $|x| = a$. In both cases one finds alternating even and odd solutions.

a) Solve the Schrödinger equation and show that a general solution $\psi(x)$ is given by

$$\psi(x) = \begin{cases} A\sin(kx) + B\cos(kx)\,, & |x| < a\,, \\ Ce^{\kappa x}\,, & x \leq -a\,, \\ De^{-\kappa x}\,, & x \geq a\,. \end{cases}$$

Determine k and κ and show that

$$k^2 + \kappa^2 = \frac{2mV_0}{\hbar^2}\,.$$

b) Since the potential is symmetric with respect to $x = 0$ we can assume that the solutions $\psi(x)$ are either even or odd. Show that the even solutions satisfy the relation

$$k\tan(ka) = \kappa\,,$$

while the odd solutions satisfy

$$k\cot(ka) = -\kappa\,.$$

Hint: Require $\psi(x)$ and $\psi'(x)$ to be continuous at $x = \pm a$.

c) Describe how the transcendental equation for k (or κ) can be solved graphically.

d) Show that there is always at least one even solution and that there is an odd solution only if $V_0 \geq \pi^2\hbar^2/8ma^2$. What is E for $V_0 = \pi^2\hbar^2/8ma^2$?