Universität Regensburg, Fakultät für Physik
Prof. Dr. Tilo Wettig WS 2023/24

Übungen zur Quantenmechanik für LA und Nanoscience
Blatt 5 (für die Übungen in der Woche 20.-24.11.)

---

1 A sequence of measurements

Suppose an operator A has two normalized eigenstates $|\psi_1\rangle$ and $|\psi_2\rangle$ with eigenvalues a_1 and a_2, respectively. Similarly, an operator B has two normalized eigenstates $|\varphi_1\rangle$ and $|\varphi_2\rangle$ with eigenvalues b_1 and b_2. Assume $a_1 \neq a_2$ and $b_1 \neq b_2$. The eigenstates satisfy the relations

$$|\psi_1\rangle = \frac{1}{5}(3|\varphi_1\rangle + 4|\varphi_2\rangle)\,,$$
$$|\psi_2\rangle = \frac{1}{5}(4|\varphi_1\rangle - 3|\varphi_2\rangle)\,.$$

a) A measurement of the observable A yields the result a_1. What is the state of the system immediately after the measurement?

b) If we now measure the observable B, what are the possible results? What are the corresponding probabilities?

c) After the measurement of B we now measure A again. What is the probability to obtain the result a_1?

2 Nonrelativistic limit of the Klein-Gordon equation

Consider the general solution

$$\psi(\vec{x},t) = \int d^3k\, A(\vec{k})e^{i(\vec{k}\cdot\vec{x}-\omega t)} \qquad (*)$$

of the free Klein-Gordon equation

$$\frac{1}{c^2}\frac{\partial^2}{\partial t^2}\psi(\vec{x},t) - \vec{\nabla}^2\psi(\vec{x},t) + \left(\frac{mc}{\hbar}\right)^2\psi(\vec{x},t) = 0\,.$$

a) Derive the corresponding dispersion relation $\hbar^2\omega^2 = m^2c^4 + \hbar^2c^2k^2$, where $k = |\vec{k}|$. What is its physical interpretation?

b) Assume that $A(\vec{k})$ falls off sufficiently rapidly for large k so that only values of k with $\hbar k/mc \ll 1$ contribute to the integral $(*)$. Using this assumption, expand $\omega(k)$ up to order k^2 and show that the solution of the Klein-Gordon equation is given by

$$\psi(\vec{x},t) = e^{-i\omega_0 t}\varphi(\vec{x},t) \quad \text{with} \quad \varphi(\vec{x},t) = \int d^3k\, A(\vec{k})e^{i(\vec{k}\cdot\vec{x}-\omega_1 t)}\,,$$

where $\omega_0 = mc^2/\hbar$ and $w_1 = \hbar k^2/2m$. Why is this the nonrelativistic limit?

c) Insert the result of part b) into the Klein-Gordon equation and show that φ satisfies the free Schrödinger equation. Explain why ψ and φ are physically equivalent.

Hint: Making the same assumption as in part b), show that $|\ddot{\varphi}/\omega_0\dot{\varphi}| \ll 1$, which implies that the $\ddot{\varphi}$ term can be neglected.

3 Uncertainty relation: general case

The goal of this problem is to prove the uncertainty relation

$$\Delta A \cdot \Delta B \geq \frac{1}{2}\left|\langle i[A,B]\rangle\right| ,$$

where A and B are Hermitian operators and $\langle A\rangle = \langle\psi|A|\psi\rangle$ for an arbitrary state vector $|\psi\rangle$. ΔA is the uncertainty of the operator A, defined by

$$(\Delta A)^2 = \langle(A - \langle A\rangle)^2\rangle = \langle\psi|(A - \langle A\rangle)^2|\psi\rangle .$$

a) Define $|f\rangle = (A - \langle A\rangle)|\psi\rangle$ and $|g\rangle = (B - \langle B\rangle)|\psi\rangle$. Show that $(\Delta A)^2 = \langle f|f\rangle$ and $(\Delta B)^2 = \langle g|g\rangle$.

b) Use the Cauchy-Schwarz inequality to show that

$$(\Delta A)^2(\Delta B)^2 \geq |\langle f|g\rangle|^2 .$$

c) Show for an arbitrary complex number z that the inequality

$$|z|^2 \geq \left[\frac{1}{2i}(z - z^*)\right]^2$$

holds. Use this inequality to show that

$$(\Delta A)^2(\Delta B)^2 \geq \left[\frac{1}{2i}(\langle f|g\rangle - \langle g|f\rangle)\right]^2 .$$

d) Show that

$$\langle f|g\rangle = \langle AB\rangle - \langle A\rangle\langle B\rangle .$$

e) Use the results of the previous steps to prove the uncertainty relation.

f) Show that the equal sign in the uncertainty relation holds if $(A - \langle A\rangle)|\psi\rangle = c(B - \langle B\rangle)|\psi\rangle$ with a purely imaginary factor c.

4 Uncertainty relation: a concrete example

In the uncertainty relation of the previous problem we now choose the operators to be position and momentum operator, i.e., $A = X$ and $B = P$. In this problem we construct the wave packet that minimizes the uncertainty relation of these two operators. For simplicity, we assume that the state $|\psi\rangle$ in which we compute the uncertainty relation is chosen such that $\langle\psi|X|\psi\rangle = 0$ and $\langle\psi|P|\psi\rangle = p_0$ with p_0 real. (Why are these assumptions justified?)

a) According to problem 3f) we should have

$$(P - p_0)|\psi\rangle = cX|\psi\rangle$$

with c imaginary. Express this condition in coordinate space and solve the resulting differential equation. Write $c = i\hbar/\Delta^2$ with Δ real and derive an expression for the wave packet $\psi(x)$. Normalize the latter.

b) Compute $\psi(p)$, the (normalized) wave packet in momentum space.

c) Compute $\Delta X \cdot \Delta P$ from the results of a) and b) and compare with the uncertainty relation.