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# Übungen zur Quantenmechanik für LA und Nanoscience
## Blatt 3 (für die Übungen in der Woche 06.-10.11.)

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### 1 Rotation matrix in two dimensions

Consider the matrix

$$\Omega = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}.$$

a) Show that it is unitary.

b) Show that its eigenvalues are $e^{i\theta}$ and $e^{-i\theta}$.

c) Find the corresponding eigenvectors and show that they are orthogonal.

d) Verify that $U^\dagger \Omega U$ is a diagonal matrix, where $U$ is the matrix of eigenvectors of $\Omega$.

### 2 Commuting Hermitian matrices

By considering the commutator, show that the following Hermitian matrices may be simultaneously diagonalized,

$$Q = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 1 \end{pmatrix}, \qquad R = \begin{pmatrix} 2 & 1 & 1 \\ 1 & 0 & -1 \\ 1 & -1 & 2 \end{pmatrix}.$$

Find the eigenvectors common to both matrices and verify that under a unitary transformation to this basis, both matrices are diagonalized. Since $Q$ is degenerate and $R$ is not, you must be prudent in deciding which matrix dictates the choice of basis.

### 3 An operator identity

Show that the operator relation

$$e^{iKa}\, X\, e^{-iKa} = X + a$$

holds, where $X$ and $K$ are the operators defined in Sec. 2.7 of the lecture.
Hint: First show that $\langle k|X|k'\rangle = i\delta'(k-k')$ by inserting unit operators. Then show that $e^{iKa}Xe^{-iKa}|\psi\rangle = (X+a)|\psi\rangle$ for a generic state $|\psi\rangle$ by expanding this state in the $|k\rangle$ basis and again inserting unit operators in the appropriate places.

## 4 A quantum-mechanical wave function

A generic quantum-mechanical state $|\psi\rangle$ may be expanded in the eigenbasis of the position operator $X$,

$$|\psi\rangle = \int_{-\infty}^{+\infty} dx\, |x\rangle\langle x|\psi\rangle = \int_{-\infty}^{+\infty} dx\, |x\rangle \psi(x)\,.$$

$\psi(x)$ is called the wave function in the coordinate-space basis. As an example, we will consider the wave function

$$\psi(x) = \frac{N}{x^2 + a^2}\,,$$

where $N$ and $a$ are real and positive. Assume that the state $|\psi\rangle$ describes a single-particle system. We will now extract information about this state using the quantum-mechanical postulates.

a) What value should the constant $N$ have so that the wave function is properly normalized? What is then the probability of finding the particle between $x$ and $x + dx$?

b) Determine the expectation value $\langle X\rangle \equiv \langle\psi|X|\psi\rangle$ of the position operator $X$ in the state $|\psi\rangle$ and the associated uncertainty $\Delta X \equiv \langle\psi|(X - \langle X\rangle)^2|\psi\rangle^{1/2}$.

c) We now consider the momentum operator $P = -i\hbar\frac{d}{dx}$ and its eigenstates $|p\rangle$. In the coordinate-space basis we encounter $\psi_p(x) = \langle x|p\rangle = e^{ipx/\hbar}/\sqrt{2\pi\hbar}$, which is normalized to the $\delta$-function. Determine $\langle p|\psi\rangle$, the wave function of $|\psi\rangle$ in the momentum-space basis.

   Hint: Insert $\mathbb{1} = \int dx\, |x\rangle\langle x|$ and use contour integration to perform the integral over $x$.

d) For our state $|\psi\rangle$, compute $\langle P\rangle$ and the corresponding uncertainty $\Delta P$.

e) Verify that the uncertainty relation $\Delta X \Delta P \geq \hbar/2$ is satisfied.