

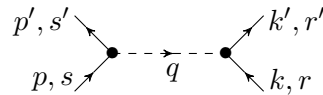
## Quantum Electrodynamics

### Problem Set 11 (for the exercises in the week of June 29)

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#### Problem 1 The Yukawa potential

Consider the scattering of *distinguishable* fermions in Yukawa theory. In this case only the diagram



contributes to  $i\mathcal{M}$  in order  $g^2$  (why?). We have shown in class that this diagram gives

$$i\mathcal{M} = (-ig^2)\bar{u}^{s'}(p')u^s(p)\frac{1}{(p' - p)^2 - m_\varphi^2}\bar{u}^{r'}(k')u^r(k).$$

- a) In the following we consider the nonrelativistic limit in which we only keep terms to lowest order in the 3-momenta ( $E_{\vec{p}} \approx m + |\vec{p}|^2/2m$ , etc.). Show that in this approximation we have

$$\begin{aligned} (p' - p)^2 &= -|\vec{p}' - \vec{p}|^2, \\ u^s(p) &= \sqrt{m} \begin{pmatrix} \xi^s \\ \xi^s \end{pmatrix}, \\ \bar{u}^{s'}(p')u^s(p) &= 2m\delta^{ss'} \end{aligned}$$

and thus

$$i\mathcal{M} = \frac{ig^2}{|\vec{p}' - \vec{p}|^2 + m_\varphi^2} 2m\delta^{ss'} 2m\delta^{rr'}.$$

- b) For scattering from a static potential it can be shown that

$$\langle \vec{p}' | iT | \vec{p} \rangle = 2\pi\delta(E_{\vec{p}'} - E_{\vec{p}}) i\mathcal{M}.$$

Furthermore, the Born approximation to the scattering amplitude in nonrelativistic quantum mechanics reads

$$\langle \vec{p}' | iT | \vec{p} \rangle = -iV(\vec{q})(2\pi)\delta(E_{\vec{p}'} - E_{\vec{p}})$$

with  $\vec{q} = \vec{p}' - \vec{p}$ . Show that we obtain a potential (in momentum space)

$$V(\vec{q}) = -\frac{g^2}{|\vec{q}|^2 + m_\varphi^2}$$

for the Yukawa interaction (modulo differences in normalization conventions).

- c) Fourier transform this result to coordinate space to show that the Yukawa potential is

$$V(r) = -\frac{g^2}{4\pi r} e^{-m_\varphi r}.$$

This potential is *attractive*.



We conclude that the exchange of a vector particle leads to repulsion between like charges and attraction between unlike charges. This explains at a fundamental level what you have learned in school.

f) Bonus question: How would the analysis above have to be amended

- if the two incoming particles are antiparticles of each other (e.g.,  $e^+e^- \rightarrow e^+e^-$ ) or
- if the scattered fermions are indistinguishable (e.g.,  $e^-e^- \rightarrow e^-e^-$ )?

### Problem 3 Some useful identities (bonus problem)

Prove the following formulas that were stated without proof in class.

a) Trace identities for  $\gamma$ -matrices:

$$\begin{aligned}\text{tr}(\mathbb{1}) &= 4 \\ \text{tr}(\text{odd number of } \gamma\text{'s}) &= 0 \\ \text{tr}(\gamma^\mu \gamma^\nu) &= 4g^{\mu\nu} \\ \text{tr}(\gamma^\mu \gamma^\nu \gamma^\rho \gamma^\sigma) &= 4(g^{\mu\nu} g^{\rho\sigma} - g^{\mu\rho} g^{\nu\sigma} + g^{\mu\sigma} g^{\nu\rho}) \\ \text{tr}(\gamma^5) &= 0 \\ \text{tr}(\gamma^\mu \gamma^\nu \gamma^5) &= 0 \\ \text{tr}(\gamma^\mu \gamma^\nu \gamma^\rho \gamma^\sigma \gamma^5) &= -4i\varepsilon^{\mu\nu\rho\sigma} \\ \text{tr}(\gamma^\mu \gamma^\nu \gamma^\rho \gamma^\sigma \dots) &= \text{tr}(\dots \gamma^\sigma \gamma^\rho \gamma^\nu \gamma^\mu)\end{aligned}$$

For the last identity, insert  $C^2 = \mathbb{1}$  between  $\gamma$ 's, where  $C = \gamma^0 \gamma^2$  and  $C\gamma^\mu C = -(\gamma^\mu)^T$ .

b) Contractions of  $\gamma$ -matrices:

$$\begin{aligned}\gamma^\mu \gamma_\mu &= 4\mathbb{1} \\ \gamma^\mu \gamma^\nu \gamma_\mu &= -2\gamma^\nu \\ \gamma^\mu \gamma^\nu \gamma^\rho \gamma_\mu &= 4g^{\nu\rho} \mathbb{1} \\ \gamma^\mu \gamma^\nu \gamma^\rho \gamma^\sigma \gamma_\mu &= -2\gamma^\sigma \gamma^\rho \gamma^\nu\end{aligned}$$

c) Contractions of  $\varepsilon$ :

$$\begin{aligned}\varepsilon^{\alpha\beta\mu\nu} \varepsilon_{\alpha\beta\rho\sigma} &= -2(\delta^\mu_\rho \delta^\nu_\sigma - \delta^\mu_\sigma \delta^\nu_\rho) \\ \varepsilon^{\alpha\beta\gamma\mu} \varepsilon_{\alpha\beta\gamma\nu} &= -6\delta^\mu_\nu \\ \varepsilon^{\alpha\beta\gamma\delta} \varepsilon_{\alpha\beta\gamma\delta} &= -24\end{aligned}$$

d) Write a computer program in the language of your choice (e.g., Octave or Python) to verify the identities numerically.