

Quantum Electrodynamics

Problem Set 9 (for the exercises in the week of June 15)

Problem 1 Chiral transformations

A chiral transformation of a spinor is defined by

$$\psi \rightarrow e^{i\alpha\gamma^5} \psi. \quad (1)$$

- a) How does the conjugate spinor $\bar{\psi}$ transform under (1)? Recall $\{\gamma^\mu, \gamma^5\} = 0$.
- b) How does the object $V^\mu = \bar{\psi}\gamma^\mu\psi$ transform under (1)?
- c) Show that the Dirac Lagrangian $\mathcal{L} = \bar{\psi}(i\gamma^\mu\partial_\mu - m)\psi$ is invariant under (1) for $m = 0$ and that it is not invariant if $m \neq 0$.
- d) Show that for $m = 0$ the Noether current is given by $j_5^\mu = \bar{\psi}\gamma^\mu\gamma^5\psi$ and that for $m \neq 0$ it satisfies $\partial_\mu j_5^\mu = 2im\bar{\psi}\gamma^5\psi$.

Problem 2 Fierz transformations

In the lecture we have discussed Dirac field bilinears such as $\bar{\psi}\gamma^\mu\psi$. They can be generalized to objects of the form

$$\bar{u}_1\Gamma^A u_2,$$

where the u_i are 4-component Dirac spinors and Γ^A is any of the following 16 ($= 1 + 4 + 6 + 4 + 1$) combinations of Dirac matrices,

$$\mathbb{1}, \quad \gamma^\mu, \quad \sigma^{\mu\nu} = \frac{i}{2}[\gamma^\mu, \gamma^\nu] \quad (\mu < \nu), \quad \gamma^\mu\gamma^5, \quad \gamma^5.$$

For practical calculations it is sometimes helpful to use rearrangement formulas for products of such bilinears. The general Fierz identity (or Fierz transformation) is given by the equation

$$(\bar{u}_1\Gamma^A u_2) (\bar{u}_3\Gamma^B u_4) = \sum_{C,D} C_{CD}^{AB} (\bar{u}_1\Gamma^C u_4) (\bar{u}_3\Gamma^D u_2)$$

with coefficients C_{CD}^{AB} to be determined.

- a) Note that the Γ^A form a complete orthogonal basis for the space of 4×4 matrices, where orthogonality is defined with respect to a scalar product given by the trace. Normalize the 16 matrices Γ^A to the convention

$$\text{tr}(\Gamma^A\Gamma^B) = 4\delta^{AB}.$$

This gives $\Gamma^A = \{\mathbb{1}, \gamma^0, i\gamma^i, \dots\}$. Write down all 16 elements of this set.

b) Using the completeness of the 16 Γ^A matrices, show that

$$C_{CD}^{AB} = \frac{1}{16} \text{tr}(\Gamma^C \Gamma^A \Gamma^D \Gamma^B).$$

Hint: The completeness implies that any 4×4 matrix can be written as a linear combination of the Γ^A . In particular, we can write

$$u_i \bar{u}_j = \sum_A c_A^{ij} \Gamma^A$$

with some coefficients c_A^{ij} . Use the cyclic property of the trace to show that $c_A^{ij} = \frac{1}{4}(\bar{u}_j \Gamma^A u_i)$. This result can be used to derive C_{CD}^{AB} .

c) Work out explicitly the Fierz transformation laws for the products $(\bar{u}_1 u_2)(\bar{u}_3 u_4)$ and $(\bar{u}_1 \gamma^\mu u_2)(\bar{u}_3 \gamma_\mu u_4)$.