Universität Regensburg, Institut für Theoretische Physik SS 2026
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Quantum Electrodynamics

Problem Set 8 (for the exercises in the week of June 8)

Problem 1 Particle creation by a classical source, Part II

Recall from Problem 3.3 that the creation of Klein-Gordon particles by a classical source can be described by the Hamiltonian

$$H = H_0 + \int d^3x\,(-j(t,\vec{x})\varphi(x))\,,$$

where H_0 is the free KG-Hamiltonian, $\varphi(x)$ is the real KG-field, and $j(x)$ is a c-number real scalar function. We found that if the system is in the vacuum state before the source is turned on, the source will create a mean number of particles

$$\langle N\rangle = \int \frac{d^3p}{(2\pi)^3}\,\frac{1}{2E_{\vec{p}}}|\tilde{\jmath}(p)|^2\,,$$

where $\tilde{\jmath}(p)$ is the Fourier transform of j for 4-momenta p such that $p^2 = m^2$. In this problem we will verify that statement and extract more detailed information by using a perturbation expansion in the strength of the source. Before you start, read the hints at the end of the problem.

a) Show that the probability that the source creates no particles at all is given by

$$P(0) = \lim_{T\to\infty}\big|\langle 0|U(T,-T)|0\rangle\big|^2 = \Big|\langle 0|T\big\{\exp\big[i\int d^4x\,j(x)\varphi_I(x)\big]\big\}|0\rangle\Big|^2\,.$$

b) Evaluate the term in $P(0)$ of order j^2, and show that $P(0) = 1-\lambda+\mathcal{O}(j^4)$, where λ equals the expression given above for $\langle N\rangle$.

c) Represent the term computed in part b) as a Feynman diagram, e.g., $\lambda = \otimes\!\!-\!\!\!-\!\!\!-\!\!\otimes = \int d^4x\,j(x)\int d^4y\,j(y)D_F(x-y)$, where the line represents a Feynman propagator and a circle represents an integration over space-time including the source. Now represent the whole perturbation series for $P(0)$ in terms of Feynman diagrams. Show that this series exponentiates, so that it can be summed exactly: $P(0) = e^{-\lambda}$.

d) Compute the probability density for the source to create one particle of momentum $\vec{k}$ and then integrate over $\vec{k}$ to get the probability to create one particle. Perform this computation first to $\mathcal{O}(j)$ and then to all orders, using the trick of part c) to sum the series.

   It is useful to introduce the Feynman diagram $\alpha(x) = \underset{x}{\bullet}\!\!-\!\!\!-\!\!\!-\!\!\otimes = \int d^4y\,j(y)D_F(x-y)$.

e) Show that the probability of producing n particles is given by the Poisson distribution

$$P(n) = \frac{1}{n!}\lambda^n e^{-\lambda}\,.$$

f) Prove the following facts about the Poisson distribution:

$$\sum_{n=0}^{\infty} P(n) = 1\,,\qquad \langle N\rangle = \sum_{n=0}^{\infty} n\,P(n) = \lambda\,.$$

   The first identity means that the $P(n)$'s are properly normalized probabilities, while the second confirms our proposal for $\langle N\rangle$. Compute the mean square fluctuation $\big\langle (N-\langle N\rangle)^2\big\rangle$.

Hint 1: Since $j(x)$ is real we have $\tilde{\jmath}(-p) = \tilde{\jmath}(p)^*$.
Hint 2: The identity operator can be written as a sum of projectors onto the vacuum state and all n-particle states,

$$\mathbb{1} = \mathbb{1}_{\text{vacuum}} + \mathbb{1}_{\text{1-particle}} + \mathbb{1}_{\text{2-particle}} + \dots$$

with $\mathbb{1}_{\text{vacuum}} = |0\rangle\langle 0|$ and

$$\mathbb{1}_{n\text{-particle}} = \frac{1}{n!}\int \frac{d^3k_1}{(2\pi)^3}\cdots\frac{d^3k_n}{(2\pi)^3}\frac{1}{2E_{\vec{k}_1}\cdots 2E_{\vec{k}_n}}\,|\vec{k}_1\cdots\vec{k}_n\rangle\langle\vec{k}_1\cdots\vec{k}_n|\,.$$

The n-particle states are defined by

$$|\vec{k}_1\cdots\vec{k}_n\rangle = \sqrt{2E_{\vec{k}_1}\cdots 2E_{\vec{k}_n}}\,a^\dagger_{\vec{k}_1}\cdots a^\dagger_{\vec{k}_n}|0\rangle$$

and satisfy

$$\langle\vec{k}_1\cdots\vec{k}_n|\vec{p}_1\cdots\vec{p}_n\rangle = (2\pi)^{3n}2E_{\vec{k}_1}\cdots 2E_{\vec{k}_n}\sum_\sigma \delta^{(3)}(\vec{k}_1-\vec{p}_{\sigma_1})\cdots\delta^{(3)}(\vec{k}_n-\vec{p}_{\sigma_n})\,,$$

where the sum is over all permutations of $1,\dots,n$.
Hint 3: Express the n-particle states in terms of the KG-field acting on the vacuum, e.g.,

$$\langle\vec{k}| = 2E_{\vec{k}}\int d^3x\, e^{ik\cdot x}\langle 0|\varphi_I^+(x)$$

for the 1-particle states. Choose x^0 to be larger than the time at which the source is turned off. This allows you to pull $\varphi_I^+(x)$ inside the time-ordering operator.

Remark: A somewhat simpler strategy to solve this problem is to work with the ladder operators. (However, if you did it this way, you wouldn't get any practice with Feynman diagrams.) You can express φ_I in terms of ladder operators and write $-i\int dt\, H_I(t) = A + B$, where A and B contain only creation and annihilation operators, respectively. The commutator of A and B is $[A,B] = \lambda$. Since $[A,[A,B]] = [B,[A,B]] = 0$, the BCH formula reads

$$e^A e^B = e^{A+B+\frac{1}{2}[A,B]}\,.$$

This can be used to eliminate the annihilation operators from the problem, since $e^B|0\rangle = 1$. The result of part e) then follows relatively straightforwardly.