Universität Regensburg, Institut für Theoretische Physik SS 2026
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Quantum Electrodynamics
Problem Set 6 (for the exercises in the week of May 25)

There is no teaching on May 26 because of the holiday weekend. If you are in the Tuesday group, please try to attend the Wednesday group or email me your solutions.

Problem 1 Time reversal

a) We have seen in class that T reverses three-momentum and spin, so we need to learn how to deal with spin flips. Let ξ^s ($s = 1, 2$) be a two-component spinor. We claim that the flipped spinor is given by

$$\xi^{-s} := -i\sigma^2(\xi^s)^* \,,$$

where σ^2 is the second Pauli matrix. To prove this, show that $\vec{\sigma}\sigma^2 = \sigma^2(-\vec{\sigma}^*)$ and use this identity to show that, if a two-component spinor ξ satisfies $(\hat{n} \cdot \vec{\sigma})\xi = +\xi$ for some axis $\hat{n}$, then $(\hat{n} \cdot \vec{\sigma})(-i\sigma^2\xi^*) = -(-i\sigma^2\xi^*)$.

b) To deal with momentum reversal, define $\tilde{p} = (p^0, -\vec{p})$ and show that

$$\sqrt{\tilde{p} \cdot \sigma}\, \sigma^2 = \sigma^2 \sqrt{p \cdot \sigma^*} \,, \qquad \sqrt{\tilde{p} \cdot \bar{\sigma}}\, \sigma^2 = \sigma^2 \sqrt{p \cdot \bar{\sigma}^*} \,.$$

Hint: $\sqrt{p \cdot \sigma}$ is a 2×2 matrix, so it can be written as a linear combination of $\mathbb{1}_2$ and the three Pauli matrices, $\sqrt{p \cdot \sigma} = \alpha_0 \mathbb{1}_2 + \alpha_i \sigma^i$. The coefficients α_μ were given in Sec. 3.3 of the lecture.

c) In Problem 5.1 explicit expressions for u and v were given,

$$u^s(p) = \begin{pmatrix} \sqrt{p \cdot \sigma}\, \xi^s \\ \sqrt{p \cdot \bar{\sigma}}\, \xi^s \end{pmatrix} \qquad \text{and} \qquad v^s(p) = \begin{pmatrix} \sqrt{p \cdot \sigma}\, \eta^s \\ -\sqrt{p \cdot \bar{\sigma}}\, \eta^s \end{pmatrix} \tag{1}$$

with $s = 1, 2$, $\xi^{r\dagger}\xi^s = \delta^{rs}$ and $\eta^{r\dagger}\eta^s = \delta^{rs}$. Using these expressions, show that the Dirac spinors with reversed momentum and spin are given by

$$u^{-s}(\tilde{p}) = -\gamma^1\gamma^3[u^s(p)]^* \,, \qquad v^{-s}(\tilde{p}) = -\gamma^1\gamma^3[v^s(p)]^* \,.$$

d) If in part a) we choose $\xi^1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and $\xi^2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$, then $\xi^{-1} = \xi^2$ and $\xi^{-2} = -\xi^1$ (this does not result in any loss of generality since the axis $\hat{n}$ is arbitrary). In analogy, we define

$$a_{\vec{p}}^{-1} = a_{\vec{p}}^2 \,, \qquad a_{\vec{p}}^{-2} = -a_{\vec{p}}^1$$

and similarly for $b_{\vec{p}}^s$. The action of T on the fermion annihilation operators is defined by

$$T a_{\vec{p}}^s T^{-1} = a_{-\vec{p}}^{-s} \,, \qquad T b_{\vec{p}}^s T^{-1} = b_{-\vec{p}}^{-s} \,,$$

i.e., momentum and spin are reversed as required. Show that the Dirac field and the various field bilinears transform under T as follows,

$$T\psi(t, \vec{x})T^{-1} = \gamma^1\gamma^3\psi(-t, \vec{x})$$

$$
\begin{aligned}
T\bar{\psi}(t,\vec{x})T^{-1} &= \bar{\psi}(-t,\vec{x})(-\gamma^1\gamma^3) \\
T\bar{\psi}\psi T^{-1} &\to \bar{\psi}\psi \\
Ti\bar{\psi}\gamma^5\psi T^{-1} &\to -i\bar{\psi}\gamma^5\psi \\
T\bar{\psi}\gamma^\mu\psi T^{-1} &\to \begin{cases} +\bar{\psi}\gamma^\mu\psi & \text{for } \mu=0 \\ -\bar{\psi}\gamma^\mu\psi & \text{for } \mu=1,2,3 \end{cases} \\
T\bar{\psi}\gamma^\mu\gamma^5\psi T^{-1} &\to \begin{cases} +\bar{\psi}\gamma^\mu\gamma^5\psi & \text{for } \mu=0 \\ -\bar{\psi}\gamma^\mu\gamma^5\psi & \text{for } \mu=1,2,3 \end{cases} \\
Ti\bar{\psi}[\gamma^\mu,\gamma^\nu]\psi T^{-1} &\to \begin{cases} -i\bar{\psi}[\gamma^\mu,\gamma^\nu]\psi & \text{for } \mu\nu=ij \\ +i\bar{\psi}[\gamma^\mu,\gamma^\nu]\psi & \text{for } \mu\nu=0i \text{ or } i0 \end{cases}
\end{aligned}
$$

Hint: Recall that T is an antiunitary operator that involves complex conjugation. The condition for an operator A to be antiunitary is

$$\langle A\alpha|A\beta\rangle = \langle\alpha|\beta\rangle^*$$

for all α, β. The adjoint of an antiunitary operator is defined by

$$\langle\alpha|A^\dagger\beta\rangle \equiv \langle A\alpha|\beta\rangle^* = \langle\beta|A\alpha\rangle\,,$$

which implies $A^\dagger = A^{-1}$.

Problem 2 Charge conjugation

In Eq. (1) the choice of ξ and η is arbitrary, subject to the constraints $\xi^{r\dagger}\xi^s = \delta^{rs}$ and $\eta^{r\dagger}\eta^s = \delta^{rs}$. You have shown in Problem 4.4 that for antifermions the spin is reversed with respect to the fermions. Therefore it is sensible to choose $\eta^s = \xi^{-s}$ with ξ^{-s} defined in Problem 1a), i.e.,

$$v^s(p) = \begin{pmatrix} \sqrt{p\cdot\sigma}\,\xi^{-s} \\ -\sqrt{p\cdot\bar{\sigma}}\,\xi^{-s} \end{pmatrix}.$$

a) Using this convention and the results of Problem 1b), show that

$$u^s(p) = -i\gamma^2[v^s(p)]^*\,, \qquad v^s(p) = -i\gamma^2[u^s(p)]^*\,.$$

b) Using

$$Ca^s_{\vec{p}}C^{-1} = b^s_{\vec{p}}\,, \qquad Cb^s_{\vec{p}}C^{-1} = a^s_{\vec{p}}\,,$$

show that the Dirac field and the various field bilinears transform under C as follows,

$$
\begin{aligned}
C\psi C^{-1} &= -i\gamma^2\psi^* = -i(\bar{\psi}\gamma^0\gamma^2)^T \\
C\bar{\psi}C^{-1} &= -i(\gamma^0\gamma^2\psi)^T \\
C\bar{\psi}\psi C^{-1} &= \bar{\psi}\psi \\
Ci\bar{\psi}\gamma^5\psi C^{-1} &= i\bar{\psi}\gamma^5\psi \\
C\bar{\psi}\gamma^\mu\psi C^{-1} &= -\bar{\psi}\gamma^\mu\psi \\
C\bar{\psi}\gamma^\mu\gamma^5\psi C^{-1} &= \bar{\psi}\gamma^\mu\gamma^5\psi \\
Ci\bar{\psi}[\gamma^\mu,\gamma^\nu]\psi C^{-1} &= -i\bar{\psi}[\gamma^\mu,\gamma^\nu]\psi
\end{aligned}
$$

Problem 3 The free electromagnetic field

a) Derive the Hamiltonian of the free electromagnetic field,

$$H = \int d^3x[\pi^\mu(x)\dot{A}_\mu(x) - \mathcal{L}] = \int \frac{d^3p}{(2\pi)^3} \sum_{r=0}^{3} \zeta_r E_{\vec{p}}\, a_{\vec{p}}^{r\dagger} a_{\vec{p}}^{r}\,,$$

using $\mathcal{L} = -\frac{1}{2}(\partial_\mu A_\nu)(\partial^\mu A^\nu)$. You can either use the expansion of the field in terms of ladder operators or find a shortcut by comparing the Lagrangian to that of the Klein-Gordon field.

b) Using the special choice for the polarization vectors $\varepsilon_r(p)$ given at the end of Sec. 4.1 of the lecture, show that the Gupta-Bleuler condition, $\partial_\mu A^{\mu+}(x)|\Psi\rangle = 0$, leads to the condition

$$(a_{\vec{p}}^3 - a_{\vec{p}}^0)|\Psi\rangle = 0 \quad \text{for all } \vec{p}.$$

c) Using the expression for H from part a), show that the condition you derived in part b) implies that only transverse photons contribute to the energy, i.e.,

$$\langle\Psi|H|\Psi\rangle = \langle\Psi| \int \frac{d^3p}{(2\pi)^3} \sum_{r=1}^{2} E_{\vec{p}}\, a_{\vec{p}}^{r\dagger} a_{\vec{p}}^{r}|\Psi\rangle\,.$$