Universität Regensburg, Institut für Theoretische Physik
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Quantum Electrodynamics

Problem Set 5 (for the exercises in the week of May 18)

Problem 1 Solutions of the free Dirac equation

We have seen in class that the free Dirac equation, $(i\partial\!\!\!/ - m)\psi(x) = 0$, has solutions with positive and negative frequency,

$$\psi(x) = u(p)e^{-ip\cdot x} \quad \text{and} \quad \psi(x) = v(p)e^{+ip\cdot x}$$

with $p^2 = m^2$ and $p^0 > 0$. The Dirac spinors u and v have the form

$$u^s(p) = \begin{pmatrix} \sqrt{p\cdot\sigma}\,\xi^s \\ \sqrt{p\cdot\bar\sigma}\,\xi^s \end{pmatrix}, \qquad v^s(p) = \begin{pmatrix} \sqrt{p\cdot\sigma}\,\eta^s \\ -\sqrt{p\cdot\bar\sigma}\,\eta^s \end{pmatrix}$$

with $s = 1, 2$. The two-spinors ξ and η are normalized by $\xi^{r\dagger}\xi^s = \delta^{rs}$ and $\eta^{r\dagger}\eta^s = \delta^{rs}$.

a) Show that $(p\cdot\sigma)(p\cdot\bar\sigma) = p^2 = m^2$, where $\sigma^\mu = (\mathbb{1}_2, \vec\sigma)$ and $\bar\sigma^\mu = (\mathbb{1}_2, -\vec\sigma)$.

b) Show that the Dirac spinors must satisfy the equations $(p\!\!\!/ - m)u(p) = 0$ and $(p\!\!\!/ + m)v(p) = 0$ and verify that these equations are indeed satisfied by $u^s(p)$ and $v^s(p)$ given above.

c) Show that

$$\begin{aligned} \bar u^r(p)u^s(p) &= 2m\delta^{rs}\,, & u^{r\dagger}(p)u^s(p) &= 2E_{\vec p}\delta^{rs}\,, \\ \bar v^r(p)v^s(p) &= -2m\delta^{rs}\,, & v^{r\dagger}(p)v^s(p) &= 2E_{\vec p}\delta^{rs}\,, \\ \bar u^r(p)v^s(p) = \bar v^r(p)u^s(p) &= 0\,, & u^{r\dagger}(\vec p)v^s(-\vec p) = v^{r\dagger}(\vec p)u^s(-\vec p) &= 0\,. \end{aligned}$$

d) Show that

$$\sum_s u^s(p)\bar u^s(p) = p\!\!\!/ + m \qquad \text{and} \qquad \sum_s v^s(p)\bar v^s(p) = p\!\!\!/ - m\,.$$

Problem 2 Effect of continuous Lorentz transformations on the Dirac field

In this problem we consider continuous (i.e., proper and orthochronous) Lorentz transformations. Let $U(\Lambda)$ be the operator that implements such a Lorentz transformation Λ on the states of the Hilbert space, i.e., $|\text{state}\rangle \to U(\Lambda)|\text{state}\rangle$.

a) Consider one-particle states of the form

$$|\vec p, s\rangle = \sqrt{2E_{\vec p}}\, a^{s\dagger}_{\vec p}|0\rangle\,.$$

We have seen in class that the inner product of two such states is Lorentz invariant. Use this fact to show that $U(\Lambda)$ is unitary.

b) Assume that the boost or rotation axis is parallel to the spin quantization axis and show that the equation above implies that $a^s_{\vec p}$ transforms like

$$U(\Lambda)\left[\sqrt{E_{\vec p}}\, a^s_{\vec p}\right] U(\Lambda^{-1}) = \sqrt{E_{\Lambda\vec p}}\, a^s_{\Lambda\vec p}\,.$$

Hint: Consider a two-particle state to motivate the $U(\Lambda^{-1})$ on the right of $a^s_{\vec p}$.

c) We already know that the field $\psi(x)$ should transform under continuous Lorentz transformations like

$$\psi(x) \to \psi'(x) = \Lambda_{\frac{1}{2}} \psi(\Lambda^{-1} x)\,. \tag{1}$$

Using

$$\psi(x) = \int \frac{d^3p}{(2\pi)^3} \frac{1}{\sqrt{2E_{\vec{p}}}} \sum_s \left(a^s_{\vec{p}} u^s(p) e^{-ip\cdot x} + b^{s\dagger}_{\vec{p}} v^s(p) e^{ip\cdot x} \right),$$

show that $U(\Lambda)$ implements the correct transformation on the field operator $\psi(x)$, i.e.,

$$U(\Lambda)\psi(x)U(\Lambda^{-1}) = \Lambda^{-1}_{\frac{1}{2}} \psi(\Lambda x)\,. \tag{2}$$

Hint: Relate $u^s(p)$ to $u^s(\Lambda p)$ and $v^s(p)$ to $v^s(\Lambda p)$ using the known transformation properties of Dirac spinors.

d) Explain the difference between the right-hand sides of Eqs. (1) and (2).

Problem 3 The quantized Dirac field

Verify the following results for the free Dirac field that were stated but not proven in class. Use the expression for $\psi(x)$ given in Problem 2c) and its Hermitian conjugate.

a) The momentum operator is

$$\vec{P} = \int d^3x\, \psi^\dagger(-i\vec{\nabla})\psi = \int \frac{d^3p}{(2\pi)^3} \sum_s \vec{p}(a^{s\dagger}_{\vec{p}} a^s_{\vec{p}} + b^{s\dagger}_{\vec{p}} b^s_{\vec{p}})\,.$$

b) The invariance of $\mathcal{L}$ under a global change of phase, $\psi(x) \to e^{i\alpha}\psi(x)$, yields the conserved Noether current

$$j^\mu = \bar{\psi}\gamma^\mu\psi$$

and the conserved charge

$$Q = \int d^3x\, j^0 = \int d^3x\, \psi^\dagger\psi = \int \frac{d^3p}{(2\pi)^3} \sum_s (a^{s\dagger}_{\vec{p}} a^s_{\vec{p}} - b^{s\dagger}_{\vec{p}} b^s_{\vec{p}})\,.$$

c) The Hamilton operator is

$$H = \int d^3x\, \bar{\psi}(-i\vec{\gamma}\cdot\vec{\nabla} + m)\psi = \int \frac{d^3p}{(2\pi)^3} \sum_s E_{\vec{p}}(a^{s\dagger}_{\vec{p}} a^s_{\vec{p}} + b^{s\dagger}_{\vec{p}} b^s_{\vec{p}})\,.$$

Hint: Use $(\not{p} - m)u = 0$, $(\not{p} + m)v = 0$, and other results of Problem 1.