

## Quantum Electrodynamics

### Problem Set 4 (for the exercises in the week of May 11)

#### Problem 1 Spinor representation of the Lorentz algebra

Assume that we have four  $n \times n$  matrices  $\gamma^\mu$  satisfying the anticommutation relations

$$\{\gamma^\mu, \gamma^\nu\} \equiv \gamma^\mu \gamma^\nu + \gamma^\nu \gamma^\mu = 2g^{\mu\nu} \mathbb{1}_n.$$

We only know these relations but not the explicit form of the  $\gamma^\mu$ . Show that the  $n \times n$  matrices

$$S^{\mu\nu} = \frac{i}{4} [\gamma^\mu, \gamma^\nu]$$

satisfy the commutation relations of the Lorentz algebra, i.e.,

$$[S^{\mu\nu}, S^{\rho\sigma}] = i(g^{\nu\rho} S^{\mu\sigma} - g^{\mu\rho} S^{\nu\sigma} - g^{\nu\sigma} S^{\mu\rho} + g^{\mu\sigma} S^{\nu\rho}).$$

Hint: First prove  $[S^{\mu\nu}, \gamma^\rho] = i(\gamma^\mu g^{\nu\rho} - \gamma^\nu g^{\mu\rho})$ .

#### Problem 2 A property of the $\gamma$ -matrices

Show that the  $\gamma$ -matrices satisfy the relation

$$\Lambda_{\frac{1}{2}}^{-1} \gamma^\mu \Lambda_{\frac{1}{2}} = \Lambda^\mu{}_\nu \gamma^\nu,$$

where

$$\begin{aligned} \Lambda_{\frac{1}{2}} &= \exp\left(-\frac{i}{2} \omega_{\mu\nu} S^{\mu\nu}\right) \quad \text{with} \quad S^{\mu\nu} = \frac{i}{4} [\gamma^\mu, \gamma^\nu], \\ \Lambda &= \exp\left(-\frac{i}{2} \omega_{\mu\nu} \mathcal{J}^{\mu\nu}\right) \quad \text{with} \quad (\mathcal{J}^{\mu\nu})_{\alpha\beta} = i(\delta_\alpha^\mu \delta_\beta^\nu - \delta_\beta^\mu \delta_\alpha^\nu). \end{aligned}$$

Hint: Substitute  $\omega_{\mu\nu} \rightarrow \alpha \omega_{\mu\nu}$  and show that the matrices  $\gamma^\mu(\alpha) = \Lambda_{1/2}^{-1}(\alpha) \gamma^\mu \Lambda_{1/2}(\alpha)$  and  $\tilde{\gamma}^\mu(\alpha) = \Lambda^\mu{}_\nu(\alpha) \gamma^\nu$  satisfy the same differential equation, i.e.,  $\partial_\alpha \gamma^\mu(\alpha) = \partial_\alpha \tilde{\gamma}^\mu(\alpha)$ . The result from the hint in Problem 1 may also be useful.

#### Problem 3 Transformation properties of $\bar{\psi}\psi$ and $\bar{\psi}\gamma^\mu\psi$

- a) We know that  $\psi$  transforms like  $\psi \rightarrow \Lambda_{\frac{1}{2}} \psi$ . Show that  $\bar{\psi} \equiv \psi^\dagger \gamma^0$  transforms like  $\bar{\psi} \rightarrow \bar{\psi} \Lambda_{\frac{1}{2}}^{-1}$ . Therefore  $\bar{\psi}\psi$  is a Lorentz scalar.

Hint: Show that  $(S^{\mu\nu})^\dagger \gamma^0 = \gamma^0 S^{\mu\nu}$ .

- b) Show that  $\bar{\psi}\gamma^\mu\psi$  is a Lorentz vector.

Hint: Use the property of the  $\gamma$ -matrices you proved in Problem 2.

### Problem 4 Orbital angular momentum and spin

In this problem, we will construct the angular momentum operator and show that the particles created by  $a_{\vec{p}}^{s\dagger}$  and  $b_{\vec{p}}^{s\dagger}$  have spin  $\frac{1}{2}$ .

- a) Since the Dirac Lagrangian is invariant under Lorentz transformations, it is invariant under rotations. Consider an infinitesimal rotation by an angle  $\theta$  about the  $z$ -axis, i.e.,  $\omega_{12} = -\omega_{21} = \theta$ , so that

$$\Lambda_{\frac{1}{2}} \approx \mathbb{1} - \frac{i}{2}\theta\Sigma^3,$$

where  $\Sigma^3 = \text{diag}(\sigma^3, \sigma^3)$ . Compute the change

$$\delta\psi = \psi'(x) - \psi(x) = \Lambda_{\frac{1}{2}}\psi(\Lambda^{-1}x) - \psi(x)$$

in the field at the point  $x$  and show that the time-component of the conserved Noether current is

$$j^0 = -i\bar{\psi}\gamma^0(x\partial_y - y\partial_x + \frac{i}{2}\Sigma^3)\psi.$$

Doing similar calculations for rotations about the  $x$ - and  $y$ -axes, one finds the angular momentum operator

$$\vec{J} = \int d^3x \psi^\dagger \left\{ \vec{x} \times (-i\vec{\nabla}) + \frac{1}{2}\vec{\Sigma} \right\} \psi.$$

- b) Express  $\psi$  and  $\psi^\dagger$  in terms of ladder operators to show that, at  $t = 0$ ,

$$J_z = \int d^3x \int \frac{d^3p}{(2\pi)^3} \frac{d^3p'}{(2\pi)^3} \frac{1}{\sqrt{2E_{\vec{p}}2E_{\vec{p}'}}} e^{i(\vec{p}-\vec{p}')\cdot\vec{x}} \\ \times \sum_{rr'} \left( a_{\vec{p}'}^{r'\dagger} u^{r'\dagger}(\vec{p}') + b_{-\vec{p}'}^{r'} v^{r'\dagger}(-\vec{p}') \right) \left( xp^2 - yp^1 + \frac{1}{2}\Sigma^3 \right) \left( a_{\vec{p}}^r u^r(\vec{p}) + b_{-\vec{p}}^{r\dagger} v^r(-\vec{p}) \right).$$

In the following we consider particles at rest. We want to show that  $a_0^{s\dagger}|0\rangle$  is an eigenstate of  $J_z$  with eigenvalue  $\pm\frac{1}{2}$ . Since  $J_z$  must annihilate the vacuum, we have  $J_z a_0^{s\dagger}|0\rangle = [J_z, a_0^{s\dagger}]|0\rangle$ . Show that the only nonzero term in the latter expression leads to

$$J_z a_0^{s\dagger}|0\rangle = \frac{1}{2m} \sum_r \left( u^{r\dagger}(0) \frac{\Sigma^3}{2} u^s(0) \right) a_0^{r\dagger}|0\rangle = \sum_r \left( \xi^{r\dagger} \frac{\sigma^3}{2} \xi^s \right) a_0^{r\dagger}|0\rangle,$$

where  $u(0)$  means  $u(p = (m, \vec{0}))$ . Choose the spinors  $\xi^r$  to be eigenstates of  $\sigma^3$  to show that we indeed have

$$J_z a_0^{s\dagger}|0\rangle = \pm \frac{1}{2} a_0^{s\dagger}|0\rangle.$$

- c) Show that for antifermions the sign is reversed, i.e.,

$$J_z b_0^{s\dagger}|0\rangle = \mp \frac{1}{2} b_0^{s\dagger}|0\rangle.$$

- d) (optional) To justify the trick used in part b), show explicitly that  $J_z$  annihilates the vacuum.