Universität Regensburg, Institut für Theoretische Physik SS 2023
Prof. Dr. Tilo Wettig

Quantum Electrodynamics
Problem Set 13 (for the exercises on July 16 and 17)

The first three problems are regular exercises. The remaining problems are optional. They cover a number of important results that were stated in class without proof. In order to gain a better understanding of radiative corrections you are strongly encouraged to do these problems as well.

Problem 1 Bremsstrahlung

a) When we discussed Bremsstrahlung we chose a reference frame in which

$$k^\mu = (k, \vec{k})\,, \quad p^\mu = E(1, \vec{v})\,, \quad p'^\mu = E(1, \vec{v}\,')$$

and introduced the function

$$I(\vec{v}, \vec{v}\,') = -\int \frac{d\Omega_{\hat{k}}}{4\pi} \left(\frac{p'}{p' \cdot \hat{k}} - \frac{p}{p \cdot \hat{k}} \right)^2 .$$

Show that $I(\vec{v}, \vec{v}\,') = 2 f_{\mathrm{IR}}(q^2)$ for all values of $q^2 < 0$, where $q = p' - p$ and

$$f_{\mathrm{IR}}(q^2) = \int_0^1 d\xi \frac{m^2 - \frac{q^2}{2}}{m^2 - q^2 \xi (1-\xi)} - 1\,.$$

Hint: Use the method of Feynman parameters to integrate the mixed term.

b) Show that in the limit $-q^2 \to \infty$ we have

$$I(\vec{v}, \vec{v}\,') \to 2 \ln \left(\frac{-q^2}{m^2} \right) .$$

Problem 2 The electron vertex function

a) If p and p' are the 4-momenta of external electrons and $q = p' - p$, show that $q^2 \le 0$.

b) The electron vertex function $\Gamma^\mu(p', p)$ can be expressed in terms of p, p', γ^μ, and constants such as m and e. Since Γ^μ transforms as a vector (why?), it can be written as

$$\Gamma^\mu = A \gamma^\mu + B(p'^\mu + p^\mu) + C(p'^\mu - p^\mu)\,.$$

Show that A, B, and C must be scalar functions and that they can only depend on q^2.

Hint: Γ^μ always appears in the combination $\bar{u}(p')\Gamma^\mu(p', p)u(p)$.

c) Use the Ward identity $q_\mu \Gamma^\mu = 0$ to show that A and B can be nonzero but that C must be zero. Same hint as in b).

Note: The Ward identity $q_\mu \Gamma^\mu = 0$ holds even though the photon is off shell. The reason for this is explained at the end of Sec. 7.5 of the lecture.

d) Use the Gordon identity from Problem 7.1 to show that Γ^μ can be written in the form

$$Z_2 \Gamma^\mu(p', p) = \gamma^\mu F_1(q^2) + \frac{i\sigma^{\mu\nu} q_\nu}{2m} F_2(q^2)\,.$$

Problem 3 Form factors, electric charge, and magnetic moment

The scattering of an electron from an external classical electromagnetic field A^{cl} can be described by adding an interaction term

$$\Delta H_{\text{int}} = \int d^3x \, e A^{\text{cl}}_\mu j^\mu$$

to the QED Hamiltonian, where $j^\mu = \bar\psi\gamma^\mu\psi$ is the electromagnetic current. In leading order, the S-matrix element for scattering from this field is

$$i\mathcal{M}(2\pi)\delta(p'^0 - p^0) = -ie\bar u(p')\gamma^\mu u(p)\,\tilde A^{\text{cl}}_\mu(p'-p)\,,$$

where $\tilde A^{\text{cl}}_\mu(q)$ is the Fourier transform of $A^{\text{cl}}_\mu(x)$. When the vertex corrections are included, this becomes

$$i\mathcal{M}(2\pi)\delta(p'^0 - p^0) = -ie\bar u(p')Z_2\Gamma^\mu(p',p)u(p)\,\tilde A^{\text{cl}}_\mu(p'-p)\,.$$

a) Consider a *nonrelativistic* electron and take A^{cl} to be a classical electrostatic potential, $A^{\text{cl}}_\mu(x) = (\varphi(\vec x), \vec 0)$. Show that

$$i\mathcal{M} = -ie\bar u(p')Z_2\Gamma^0(p',p)u(p)\,\tilde\varphi(\vec q)\,.$$

If the electrostatic field varies slowly over space, we can take the limit $\vec q \to 0$ in the spinor matrix element. Using a nonrelativistic result from Problem 11.2b), show that in this limit

$$i\mathcal{M} = -ieF_1(0)\tilde\varphi(\vec q)\,2m\xi'^\dagger\xi\,.$$

This is the Born approximation for scattering from a potential

$$V(\vec x) = eF_1(0)\varphi(\vec x)\,.$$

We thus conclude that $F_1(0)$ is the electric charge of the electron in units of e, i.e., $F_1(0) = 1$.

b) We now repeat this analysis for a static vector potential, $A^{\text{cl}}_\mu(x) = (0, \vec A^{\text{cl}}(\vec x))$, i.e.,

$$i\mathcal{M} = ie\left[\bar u(p')\left(\gamma^i F_1 + \frac{i\sigma^{i\nu}q_\nu}{2m}F_2\right)u(p)\right]\tilde A^i_{\text{cl}}(\vec q)\,. \tag{1}$$

To take the nonrelativistic limit, we need to extract the term linear in q^j in the square brackets in Eq. (1). For the F_2-term this is quite easy. First show, using the definition $\sigma^{ij} = \frac{i}{2}[\gamma^i,\gamma^j]$, that this term becomes

$$\bar u(p')\frac{i\sigma^{ij}q_j}{2m}F_2(0)u(p) = \frac{-i\varepsilon^{ijk}q^j}{2m}F_2(0)\bar u(p')\,\text{diag}(\sigma^k,\sigma^k)u(p)\,.$$

Now recall from Problem 11.1a) that in the nonrelativistic limit we have

$$u(p) \approx \sqrt{m}\begin{pmatrix}\xi\\ \xi\end{pmatrix}$$

and show that the F_2-term becomes

$$2m\xi'^\dagger\left(\frac{-i}{2m}\varepsilon^{ijk}q^j\sigma^k F_2(0)\right)\xi$$

in this limit. For the F_1-term we use the Gordon identity,

$$\bar u(p')\gamma^i u(p) = \frac{1}{2m}(p'+p)^i\bar u(p')u(p) + \frac{i}{2m}\bar u(p')\sigma^{ik}q_k u(p)\,.$$

The first term is spin-independent and thus irrelevant for the magnetic-moment interaction we are interested in (it actually corresponds to an interaction of the magnetic field with the orbital angular momentum). Therefore we do not consider it further. Use the similarity of the second term with the structure of the F_2-term to show that in the nonrelativistic limit Eq. (1) becomes

$$i\mathcal{M} = -i(2m)\, e\xi'^{\dagger} \left(\frac{-1}{2m}\sigma^k[F_1(0) + F_2(0)]\right) \xi \tilde{B}^k(\vec{q})\,,$$

where $\tilde{B}^k(\vec{q}) = i\varepsilon^{ijk} q^i \tilde{A}^j_{\rm cl}(\vec{q})$ is the Fourier transform of the magnetic field produced by $\vec{A}^{\rm cl}(\vec{x})$. This result can again be interpreted as the Born approximation for scattering from a potential that has the form of a magnetic-moment interaction,

$$V(\vec{x}) = -\langle\vec{\mu}\rangle \cdot \vec{B}(\vec{x}) \qquad \text{with} \qquad \langle\vec{\mu}\rangle = \frac{e}{m}[F_1(0) + F_2(0)]\xi'^{\dagger}\frac{\vec{\sigma}}{2}\xi\,.$$

We can write this as

$$\vec{\mu} = g\left(\frac{e}{2m}\right)\vec{S}\,,$$

where $\vec{S} = \vec{\sigma}/2$ is the electron spin and $g = 2[F_1(0) + F_2(0)] = 2 + 2F_2(0)$.

Problem 4 One-loop contribution to the electron vertex function

In class we computed the one-loop contribution to the electron vertex function but omitted some of the intermediate steps. We now fill in the gaps.

a) Using some identities from Problem 11.3b), show that

$$\begin{aligned} -ig_{\nu\rho}\bar{u}(p')(-ie\gamma^\nu)i(\not{k}' + m)\gamma^\mu i(\not{k} + m)(-ie\gamma^\rho)u(p) \\ = 2ie^2\bar{u}(p')[\not{k}\gamma^\mu\not{k}' + m^2\gamma^\mu - 2m(k + k')^\mu]u(p)\,. \end{aligned} \tag{2}$$

b) Starting from the RHS of Eq. (2) and using

$$\int \frac{d^4\ell}{(2\pi)^4}\frac{\ell^\mu}{D^3} = 0 \quad \text{and} \quad \int \frac{d^4\ell}{(2\pi)^4}\frac{\ell^\mu\ell^\nu}{D^3} = \frac{1}{4}g^{\mu\nu}\int \frac{d^4\ell}{(2\pi)^4}\frac{\ell^2}{D^3}$$

with $\ell = k + yq - zp$ and $D = \ell^2 - \Delta + i\varepsilon$ as defined in class, show that

$$\begin{aligned} \bar{u}(p')\,\delta\Gamma^\mu(p',p)\,u(p) = 2ie^2 \int \frac{d^4\ell}{(2\pi)^4}\int_0^1 dxdydz\,\delta(x+y+z-1)\frac{2}{D^3} \\ \cdot\,\bar{u}(p')\left[\gamma^\mu\left(-\frac{1}{2}\ell^2 + (1-x)(1-y)q^2 + (1-4z+z^2)m^2\right) + \frac{i\sigma^{\mu\nu}q_\nu}{2m}2m^2 z(1-z)\right]u(p)\,. \end{aligned}$$

Problem 5 Infrared divergences in the vertex function

The infrared divergence we encountered in $\delta F_1(q^2)$ can be regularized by a small photon mass μ. We found in class that the part which dominates in the $\mu \to 0$ limit is

$$\delta F_1(q^2) = \frac{\alpha}{2\pi}\int_0^1 dxdydz\,\delta(x+y+z-1)\left[\frac{m^2(1-4z+z^2) + q^2(1-x)(1-y)}{m^2(1-z)^2 - q^2xy + \mu^2 z} - \frac{m^2(1-4z+z^2)}{m^2(1-z)^2 + \mu^2 z}\right].$$

In the following we only keep divergent terms.

a) The divergence comes from the region where $z \approx 1$. Use this fact to show that

$$\delta F_1(q^2) = \frac{\alpha}{2\pi}\int_0^1 dz\int_0^{1-z} dy\left[\frac{-2m^2 + q^2}{m^2(1-z)^2 - q^2y(1-y-z) + \mu^2} - \frac{-2m^2}{m^2(1-z)^2 + \mu^2}\right].$$

b) Now change variables to $y = (1-z)\xi$ and $w = 1-z$ and show that

$$\delta F_1(q^2) = \frac{\alpha}{4\pi}\int_0^1 d\xi \left[\frac{-2m^2+q^2}{m^2-q^2\xi(1-\xi)}\ln\left(\frac{m^2-q^2\xi(1-\xi)}{\mu^2}\right)+2\ln\left(\frac{m^2}{\mu^2}\right)\right].$$

c) Show that in the limit $\mu \to 0$ this becomes

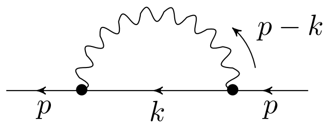
$$\delta F_1(q^2) = -\frac{\alpha}{2\pi} f_{\rm IR}(q^2)\ln\left(\frac{-q^2 \text{ or } m^2}{\mu^2}\right) \quad \text{with} \quad f_{\rm IR}(q^2) = \int_0^1 d\xi \frac{m^2-\frac{q^2}{2}}{m^2-q^2\xi(1-\xi)} - 1\,.$$

Recall from Problem 1b) that $f_{\rm IR}(q^2) \to \ln(-q^2/m^2)$ for $-q^2 \to \infty$.

d) Show that $f_{\rm IR}(q^2) > 0$ for $q^2 < 0$.

Problem 6 The electron self-energy

In class we discussed the $\mathcal{O}(\alpha)$ contribution Σ_2 to the electron self-energy, given by the diagram



In the following we compute this diagram, using the methods introduced in Sec. 7.3.2 of the lecture.

a) Starting from

$$-i\Sigma_2(p) = (-ie)^2\int\frac{d^4k}{(2\pi)^4}\,\gamma^\mu\frac{i(\not{k}+m_0)}{k^2-m_0^2+i\varepsilon}\gamma_\mu\frac{-i}{(p-k)^2-\mu^2+i\varepsilon}\,,$$

introduce a Feynman parameter, complete the square, and use symmetries to show that

$$-i\Sigma_2(p) = -e^2\int_0^1 dx\int\frac{d^4\ell}{(2\pi)^4}\,\frac{-2x\not{p}+4m_0}{(\ell^2-\Delta+i\varepsilon)^2}$$

with $\Delta = -x(1-x)p^2+x\mu^2+(1-x)m_0^2$. Here, μ is a small photon mass that was introduced to regulate the IR divergence of the integral.

b) The integral above is UV-divergent. Use Pauli-Villars regularization, perform a Wick rotation, and compute the momentum integral to show that

$$\Sigma_2(p) = \frac{\alpha}{2\pi}\int_0^1 dx\,(2m_0-x\not{p})\ln\left(\frac{x\Lambda^2}{(1-x)m_0^2+x\mu^2-x(1-x)p^2}\right),$$

where Λ is the mass of the fictitious heavy photon.

c) Show that the $\mathcal{O}(\alpha)$ mass shift of the electron is given by $\delta m = m - m_0 = \Sigma_2(\not{p}=m) \approx \Sigma_2(\not{p}=m_0)$ with

$$\begin{aligned}\delta m &= \frac{\alpha}{2\pi}m_0\int_0^1 dx\,(2-x)\ln\left(\frac{x\Lambda^2}{(1-x)^2m_0^2+x\mu^2}\right)\\ &\overset{\Lambda\to\infty}{\to} \frac{3\alpha}{4\pi}m_0\ln\left(\frac{\Lambda^2}{m_0^2}\right).\end{aligned}$$

d) Show that the $\mathcal{O}(\alpha)$ shift of Z_2 is given by

$$\delta Z_2 = \left.\frac{d\Sigma_2}{d\not{p}}\right|_{\not{p}=m} = \frac{\alpha}{2\pi}\int_0^1 dx\left[-x\ln\frac{x\Lambda^2}{(1-x)^2m^2+x\mu^2}+2(2-x)\frac{x(1-x)m^2}{(1-x)^2m^2+x\mu^2}\right].$$

e) Show that $\delta Z_2 = -I_1(0)$, where $I_1(q^2)$ is the integral introduced in Sec. 7.3.2 that leads to a correction to $F_1(q^2)$, i.e.,

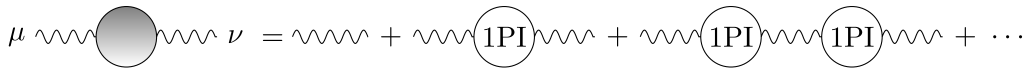
$$I_1(q^2) = \frac{\alpha}{2\pi}\int_0^1 dxdydz\,\delta(x+y+z-1)\left[\ln\frac{z\Lambda^2}{\Delta'} + \frac{(1-x)(1-y)q^2 + (1-4z+z^2)m^2}{\Delta'}\right]$$

with $\Delta' = -xyq^2 + (1-z)^2m^2 + z\mu^2$.

Hint: Use integration by parts.

Problem 7 The photon two-point function

In class we showed that the exact photon two-point function is given by

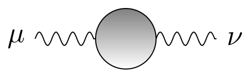


$$= \frac{-ig_{\mu\nu}}{q^2} + \frac{-ig_{\mu\rho}}{q^2}\left[i(q^2g^{\rho\sigma} - q^\rho q^\sigma)\Pi(q^2)\right]\frac{-ig_{\sigma\nu}}{q^2} + \dots\,,$$

where 1PI stands for the sum of all one-particle irreducible photon self-energy diagrams. In the following we sum this geometric series.

a) Define $\Delta^\rho_\nu = \delta^\rho_\nu - q^\rho q_\nu/q^2$ and show that $\Delta^\rho_\sigma\Delta^\sigma_\nu = \Delta^\rho_\nu$.

b) Use this result to show that



$$= \frac{-ig_{\mu\nu}}{q^2} + \frac{-ig_{\mu\rho}}{q^2}\Delta^\rho_\nu\left[\Pi(q^2) + \Pi^2(q^2) + \dots\right]$$

$$= \frac{-i}{q^2[1-\Pi(q^2)]}\left(g_{\mu\nu} - \frac{q_\mu q_\nu}{q^2}\right) + \frac{-i}{q^2}\left(\frac{q_\mu q_\nu}{q^2}\right).$$

Problem 8 The Uehling potential

In class we derived the following expression for the one-loop contribution to $\Pi(q^2)$,

$$\hat{\Pi}_2(q^2) = \Pi_2(q^2) - \Pi_2(0) = -\frac{2\alpha}{\pi}\int_0^1 dx\,x(1-x)\ln\left(\frac{m^2}{m^2 - x(1-x)q^2}\right).$$

In the following we consider the nonrelativistic limit (as in problems 11.1 and 11.2). The Coulomb potential then gets modified according to

$$V(\vec{x}) = \int\frac{d^3q}{(2\pi)^3}\,e^{i\vec{q}\cdot\vec{x}}\frac{-e^2}{|\vec{q}|^2[1-\hat{\Pi}_2(-|\vec{q}|^2)]}$$

$$= \frac{ie^2}{(2\pi)^2r}\int_{-\infty}^{\infty}dQ\,\frac{Qe^{iQr}}{Q^2+\mu^2}\left[1 + \hat{\Pi}_2(-Q^2) + \mathcal{O}(\alpha^2)\right],$$

where $Q = |\vec{q}|$ and a small photon mass μ has been introduced to regulate the Coulomb potential.

a) Analyze the analytic structure of the integrand in the complex Q plane and show that the leading contribution to $V(r)$ comes from the pole at $Q = i\mu$, resulting in $V(r) = -\alpha/r$.

b) Show that the contribution to the potential from the branch cut beginning at $Q = 2mi$ is

$$\delta V(r) = -\frac{\alpha}{r}\cdot\frac{2}{\pi}\int_{2m}^{\infty}dq\,\frac{e^{-qr}}{q}\frac{\alpha}{3}\sqrt{1-\frac{4m^2}{q^2}}\left(1+\frac{2m^2}{q^2}\right).$$

c) Show that in the limit $r \gg 1/m$ this becomes

$$\delta V(r) \approx -\frac{\alpha}{r}\cdot\frac{\alpha}{4\sqrt{\pi}}\frac{e^{-2mr}}{(mr)^{3/2}}.$$

This radiative correction to the Coulomb potential is called the Uehling potential.