Universität Regensburg, Institut für Theoretische Physik SS 2024
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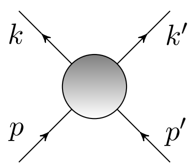
Quantum Electrodynamics

Problem Set 12 (for the exercises on July 9 and 10)

Problem 1 Mandelstam variables

Consider a generic 2-body $\to$ 2-body scattering process with the following assignments for the momenta of the incoming and outgoing particles and the corresponding definition of Mandelstam variables.



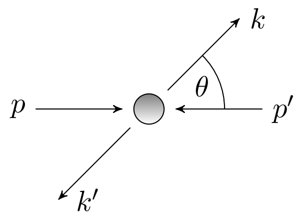
$$s = (p+p')^2 = (k+k')^2$$
$$t = (k-p)^2 = (k'-p')^2$$
$$u = (k'-p)^2 = (k-p')^2$$

a) Show that

$$s+t+u = \sum_{i=1}^{4} m_i^2\,,$$

where the m_i are the masses of the particles involved in the scattering process.

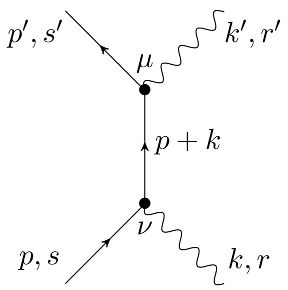
b) Consider the special case in which all four masses are equal to m. Evaluate the Mandelstam variables explicitly in the center-of-mass frame



and show how they depend on the scattering angle θ. For what values of θ (if any) do they go to zero?

Problem 2 Compton scattering I: Lorentz-invariant results

As shown in class, Compton scattering in lowest order is described by the two diagrams



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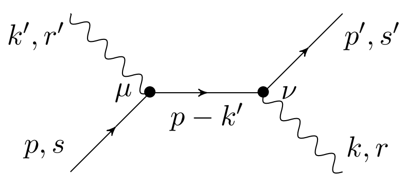


and $i\mathcal{M}$ is given by

$$i\mathcal{M} = -ie^2 \varepsilon_\mu^{r'}(k')^* \varepsilon_\nu^r(k) \bar{u}^{s'}(p') \left[\frac{\gamma^\mu \not{k} \gamma^\nu + 2\gamma^\mu p^\nu}{2p \cdot k} + \frac{-\gamma^\nu \not{k}' \gamma^\mu + 2\gamma^\nu p^\mu}{-2p \cdot k'} \right] u^s(p) \,.$$

We now derive the result for the unpolarized cross section that was stated in class without proof. We need $\frac{1}{4}\sum_{\text{spins}} |\mathcal{M}|^2$, where the sum is over the polarizations of the incoming and outgoing electrons and photons.

a) Show that this sum can be written as

$$\frac{1}{4}\sum_{\text{spins}} |\mathcal{M}|^2 = \frac{e^4}{4}\left[\frac{\mathbf{I}}{(2p\cdot k)^2} + \frac{\mathbf{II}}{(2p\cdot k)(2p\cdot k')} + \frac{\mathbf{III}}{(2p\cdot k')(2p\cdot k)} + \frac{\mathbf{IV}}{(2p\cdot k')^2}\right]$$
$$\text{with} \quad \mathbf{III} = \mathbf{II} \quad \text{and} \quad \mathbf{IV} = \mathbf{I}\,(k \to -k')\,.$$

The numerators contain complicated traces.

Hint: For the sum over electron polarizations, use the completeness relation $\sum_s u^s(p)\bar{u}^s(p) = \not{p}+m$, see Sec. 3.3 of the lecture. For the sum over photon polarizations, use the replacement rule $\sum_{\text{polarizations}} \varepsilon_\mu^* \varepsilon_\nu \to -g_{\mu\nu}$ from Sec. 6.4.1 of the lecture.

b) Using the formulas from Sec. 6.1 of the lecture, show that

$$\mathbf{I} = 16(4m^4 - 2m^2 p\cdot p' + 4m^2 p\cdot k - 2m^2 p'\cdot k + 2(p\cdot k)(p'\cdot k))\,.$$

Using the Mandelstam variables corresponding to our assignments of momenta,

$$s = (p+k)^2\,, \quad t = (p'-p)^2\,, \quad u = (k'-p)^2\,,$$

show that this can be written as

$$\mathbf{I} = 16\big(2m^4 + m^2(s-m^2) - \tfrac{1}{2}(s-m^2)(u-m^2)\big)$$

and that

$$\mathbf{IV} = 16\big(2m^4 + m^2(u-m^2) - \tfrac{1}{2}(s-m^2)(u-m^2)\big)\,.$$

c) Show similarly that

$$\mathbf{II} = \mathbf{III} = -8\big(4m^4 + m^2(s-m^2) + m^2(u-m^2)\big)\,.$$

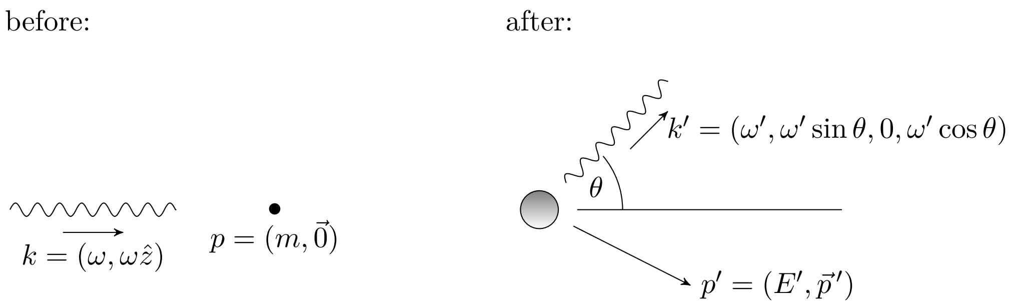
d) Put all of these results together to show that

$$\frac{1}{4}\sum_{\text{spins}} |\mathcal{M}|^2 = 2e^4\left[\frac{p\cdot k'}{p\cdot k} + \frac{p\cdot k}{p\cdot k'} + 2m^2\left(\frac{1}{p\cdot k} - \frac{1}{p\cdot k'}\right) + m^4\left(\frac{1}{p\cdot k} - \frac{1}{p\cdot k'}\right)^2\right].$$

Congratulations.

Problem 3 Compton scattering II: Cross section in the laboratory frame

We now use the result of Problem 2d) to compute the unpolarized cross section in the laboratory frame, in which the electron is initially at rest.



a) Using the conservation of four-momentum, show that

$$\frac{1}{\omega'} - \frac{1}{\omega} = \frac{1}{m}(1 - \cos\theta)\,,$$

a result we already know from quantum mechanics.

b) Show that the 2-body phase space integral in the lab frame is

$$\begin{aligned}\int d\Pi_2 &= \int \frac{d^3k'}{(2\pi)^3}\frac{1}{2\omega'}\int \frac{d^3p'}{(2\pi)^3}\frac{1}{2E'}(2\pi)^4\delta^{(4)}(k' + p' - k - p)\\ &= \frac{1}{8\pi}\int d(\cos\theta)\frac{(\omega')^2}{\omega m}\,.\end{aligned}$$

c) Using the relation between $\mathcal{M}$ and the cross section (see Sec. 5.7.2 of the lecture), show that

$$\frac{d\sigma}{d\cos\theta} = \frac{1}{2\omega}\frac{1}{2m}\cdot\frac{1}{8\pi}\frac{(\omega')^2}{\omega m}\cdot\frac{1}{4}\sum_{\text{spins}}|\mathcal{M}|^2 = \frac{\pi\alpha^2}{m^2}\left(\frac{\omega'}{\omega}\right)^2\left[\frac{\omega'}{\omega} + \frac{\omega}{\omega'} - \sin^2\theta\right]\,.$$

This is the spin-averaged Klein-Nishina formula.

d) Show that in the limit $\omega \to 0$ one obtains

$$\begin{aligned}\frac{d\sigma}{d\cos\theta} &= \frac{\pi\alpha^2}{m^2}(1 + \cos^2\theta)\,,\\ \sigma_{\text{total}} &= \frac{8\pi\alpha^2}{3m^2}\,.\end{aligned}$$

This is the Thomson cross section for the scattering of classical electromagnetic radiation by a free electron.