Universität Regensburg, Institut für Theoretische Physik SS 2024
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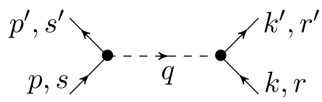
Quantum Electrodynamics

Problem Set 11 (for the exercises on July 2 and 3)

Problem 1 The Yukawa potential

Consider the scattering of *distinguishable* fermions in Yukawa theory. In this case only the diagram



contributes to $i\mathcal{M}$ in order g^2 (why?). We have shown in class that this diagram gives

$$i\mathcal{M} = (-ig^2)\bar{u}^{s'}(p')u^s(p)\frac{1}{(p'-p)^2 - m_\varphi^2}\bar{u}^{r'}(k')u^r(k)\,.$$

a) In the following we consider the nonrelativistic limit in which we only keep terms to lowest order in the 3-momenta ($E_{\vec{p}} \approx m + |\vec{p}|^2/2m$, etc.). Show that in this approximation we have

$$(p'-p)^2 = -|\vec{p}\,' - \vec{p}|^2\,,$$
$$u^s(p) = \sqrt{m}\begin{pmatrix}\xi^s\\ \xi^s\end{pmatrix},$$
$$\bar{u}^{s'}(p')u^s(p) = 2m\delta^{ss'}$$

and thus

$$i\mathcal{M} = \frac{ig^2}{|\vec{p}\,'-\vec{p}|^2 + m_\varphi^2}2m\delta^{ss'}2m\delta^{rr'}\,.$$

b) For scattering from a static potential it can be shown that

$$\langle\vec{p}\,'|iT|\vec{p}\rangle = 2\pi\delta(E_{\vec{p}} - E_{\vec{p}\,'})\,i\mathcal{M}\,.$$

Furthermore, the Born approximation to the scattering amplitude in nonrelativistic quantum mechanics reads

$$\langle\vec{p}\,'|iT|\vec{p}\rangle = -iV(\vec{q})(2\pi)\delta(E_{\vec{p}\,'} - E_{\vec{p}})$$

with $\vec{q} = \vec{p}\,' - \vec{p}$. Show that we obtain a potential (in momentum space)

$$V(\vec{q}) = -\frac{g^2}{|\vec{q}|^2 + m_\varphi^2}$$

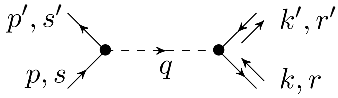
for the Yukawa interaction (modulo differences in normalization conventions).

c) Fourier transform this result to coordinate space to show that the Yukawa potential is

$$V(r) = -\frac{g^2}{4\pi r}e^{-m_\varphi r}\,.$$

This potential is *attractive*.

d) Now consider the scattering of a fermion and an antifermion, described by the diagram



Show that in the nonrelativistic limit we have

$$\bar{v}^r(k)v^{r'}(k') = -2m\delta^{rr'}$$

and modify the analysis above to show that we get the same *attractive* potential in this case.

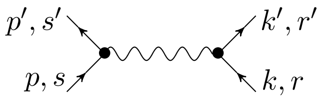
e) Finally, consider the scattering of two antifermions in the nonrelativistic limit. Show that the resulting potential is again *attractive*.

We conclude that the exchange of a scalar particle always leads to attraction between spin-$\frac{1}{2}$ particles.

Problem 2 The Coulomb potential

Here we repeat the nonrelativistic calculation of Problem 1, but now for QED instead of Yukawa theory. Use the results of Problem 1 whenever possible.

a) Show that for the scattering of two distinguishable fermions, the leading-order contribution to $i\mathcal{M}$ is

$i\mathcal{M} =$ 

$$= (-ie)^2 \bar{u}^{s'}(p')\gamma^\mu u^s(p) \frac{-ig_{\mu\nu}}{(p'-p)^2} \bar{u}^{r'}(k')\gamma^\nu u^r(k)\,.$$

b) Show that in the nonrelativistic limit we have

$$\bar{u}^{s'}(p')\gamma^0 u^s(p) = 2m\delta^{ss'}\,,$$
$$\bar{u}^{s'}(p')\gamma^i u^s(p) = 0$$

so that the expression for $i\mathcal{M}$ above becomes

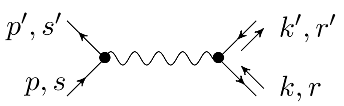
$$i\mathcal{M} = \frac{-ie^2}{|\vec{p}\,'-\vec{p}|^2} 2m\delta^{ss'} 2m\delta^{rr'}\,.$$

c) Compare with the Yukawa case in Problem 1 and show that the Coulomb potential for the present case is

$$V(r) = \frac{e^2}{4\pi r} = \frac{\alpha}{r}\,.$$

This potential is *repulsive*.

d) Now consider fermion-antifermion scattering. Show that

$i\mathcal{M} =$ 

$$= (-1)\frac{-ie^2}{|\vec{p}\,'-\vec{p}|^2} 2m\delta^{ss'} 2m\delta^{rr'}\,,$$

which leads to an *attractive* potential.

e) Show that for the scattering of two antifermions the potential is *repulsive* again.

We conclude that the exchange of a vector particle leads to repulsion between like charges and attraction between unlike charges. This explains at a fundamental level what you have learned in school.

f) Bonus question: How would the analysis above have to be amended

- if the two incoming particles are antiparticles of each other (e.g., $e^+e^- \to e^+e^-$) or
- if the scattered fermions are indistinguishable (e.g., $e^-e^- \to e^-e^-$)?

Problem 3 Some useful identities (bonus problem)

Prove the following formulas that were stated without proof in class.

a) Trace identities for γ-matrices:

$$\begin{aligned}
\mathrm{tr}(\mathbb{1}) &= 4 \\
\mathrm{tr}(\text{odd number of } \gamma\text{'s}) &= 0 \\
\mathrm{tr}(\gamma^\mu \gamma^\nu) &= 4g^{\mu\nu} \\
\mathrm{tr}(\gamma^\mu \gamma^\nu \gamma^\rho \gamma^\sigma) &= 4(g^{\mu\nu} g^{\rho\sigma} - g^{\mu\rho} g^{\nu\sigma} + g^{\mu\sigma} g^{\nu\rho}) \\
\mathrm{tr}(\gamma^5) &= 0 \\
\mathrm{tr}(\gamma^\mu \gamma^\nu \gamma^5) &= 0 \\
\mathrm{tr}(\gamma^\mu \gamma^\nu \gamma^\rho \gamma^\sigma \gamma^5) &= -4i\varepsilon^{\mu\nu\rho\sigma} \\
\mathrm{tr}(\gamma^\mu \gamma^\nu \gamma^\rho \gamma^\sigma \cdots) &= \mathrm{tr}(\cdots \gamma^\sigma \gamma^\rho \gamma^\nu \gamma^\mu)
\end{aligned}$$

For the last identity, insert $C^2 = \mathbb{1}$ between γ's, where $C = \gamma^0\gamma^2$ and $C\gamma^\mu C = -(\gamma^\mu)^T$.

b) Contractions of γ-matrices:

$$\begin{aligned}
\gamma^\mu \gamma_\mu &= 4\mathbb{1} \\
\gamma^\mu \gamma^\nu \gamma_\mu &= -2\gamma^\nu \\
\gamma^\mu \gamma^\nu \gamma^\rho \gamma_\mu &= 4g^{\nu\rho}\mathbb{1} \\
\gamma^\mu \gamma^\nu \gamma^\rho \gamma^\sigma \gamma_\mu &= -2\gamma^\sigma \gamma^\rho \gamma^\nu
\end{aligned}$$

c) Contractions of ε:

$$\begin{aligned}
\varepsilon^{\alpha\beta\mu\nu} \varepsilon_{\alpha\beta\rho\sigma} &= -2(\delta^\mu{}_\rho \delta^\nu{}_\sigma - \delta^\mu{}_\sigma \delta^\nu{}_\rho) \\
\varepsilon^{\alpha\beta\gamma\mu} \varepsilon_{\alpha\beta\gamma\nu} &= -6\delta^\mu{}_\nu \\
\varepsilon^{\alpha\beta\gamma\delta} \varepsilon_{\alpha\beta\gamma\delta} &= -24
\end{aligned}$$

d) Write a computer program in the language of your choice (e.g., Octave or Python) to verify the identities numerically.