Universität Regensburg, Institut für Theoretische Physik
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SS 2024

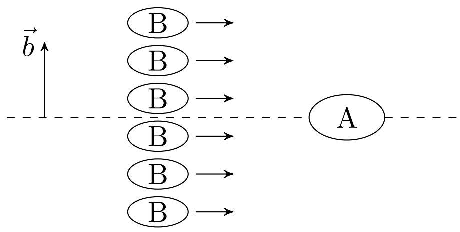
Quantum Electrodynamics
Problem Set 10 (for the exercises on June 25 and 26)

Problem 1 Cross sections in terms of the S-matrix

In this problem we will derive the formula for the differential cross section $d\sigma$ in terms of the invariant matrix element $\mathcal{M}$ that was stated without proof in class.
Since the wave packets describing the in- and out-states are localized in space, they can be constructed independently at different locations. Therefore the initial state can be written in terms of in-states of definite momentum as

$$|\varphi_A\varphi_B\rangle_{\text{in}} = \int \frac{d^3k_A}{(2\pi)^3} \int \frac{d^3k_B}{(2\pi)^3} \frac{\varphi_A(\vec{k}_A)\varphi_B(\vec{k}_B)e^{-i\vec{b}\cdot\vec{k}_B}}{\sqrt{2E_A 2E_B}} |\vec{k}_A\vec{k}_B\rangle_{\text{in}} \,.$$

The factor $e^{-i\vec{b}\cdot\vec{k}_B}$ accounts for the fact that in a scattering experiment the two beams don't always collide head-on but can be displaced in the transverse direction by an amount $\vec{b}$ (called the "impact parameter"), see figure.



Similarly, the final state can be written as

$$_{\text{out}}\langle\varphi_1\cdots\varphi_n| = \left(\prod_f \int \frac{d^3p_f}{(2\pi)^3} \frac{\varphi_f(\vec{p}_f)}{\sqrt{2E_f}}\right) {}_{\text{out}}\langle\vec{p}_1\cdots\vec{p}_n| \,.$$

a) Instead of working with out-states described by wave packets it is more convenient to use out-states of definite momentum as the final states in the scattering amplitude. Assume that the wave packets are sufficiently narrow in momentum space so that the detectors mainly measure momentum and do not resolve positions at the level of de Broglie wavelengths. Show that under this assumption the probability for the initial state $|\varphi_A\varphi_B\rangle_{\text{in}}$ to scatter into a final state of n particles with momenta in the small region $d^3p_1\cdots d^3p_n$ is

$$P(AB\to 1\cdots n) = \left(\prod_f \frac{d^3p_f}{(2\pi)^3}\frac{1}{2E_f}\right) |{}_{\text{out}}\langle\vec{p}_1\cdots\vec{p}_n|\varphi_A\varphi_B\rangle_{\text{in}}|^2 \,,$$

i.e., the normalization factors are included *after* squaring the amplitude. Note that P can depend on the impact parameter $\vec{b}$.

b) Assume that we have a single target particle (A) and many incident particles (B) with different impact parameters. Show that the number of scattering events is

$$dN = \int d^2b\, n_B(\vec{b})P(\vec{b}) \,,$$

where $n_B(\vec{b})$ is the number density per unit area of the B particles. If n_B is constant over the range of integration, show that the cross section is

$$d\sigma = \int d^2b \, P(\vec{b}) \, .$$

c) The formula for $d\sigma$ now becomes

$$d\sigma = \left(\prod_f \frac{d^3p_f}{(2\pi)^3} \frac{1}{2E_f}\right) \int d^2b \left(\prod_{i=A,B} \int \frac{d^3k_i}{(2\pi)^3} \frac{\varphi_i(\vec{k}_i)}{\sqrt{2E_i}} \int \frac{d^3k_i'}{(2\pi)^3} \frac{\varphi_i^*(\vec{k}_i')}{\sqrt{2E_i'}}\right)$$
$$\cdot \, e^{i\vec{b}\cdot(\vec{k}_B' - \vec{k}_B)} \cdot {}_{\text{out}}\langle\{\vec{p}_f\}|\{\vec{k}_i\}\rangle_{\text{in}} \cdot {}_{\text{out}}\langle\{\vec{p}_f\}|\{\vec{k}_i'\}\rangle_{\text{in}}^* \, .$$

Express the last two factors in terms of $\mathcal{M}$ as defined in class (you can drop the $\mathbb{1}$ in $S = \mathbb{1} + iT$ since we are not interested in the trivial case of forward scattering without interactions). Then use one of the $\delta^{(4)}$-functions and another $\delta^{(2)}$-function (from the integration over d^2b) to perform the six integrals over d^3k_i'.
Since the initial wave packets are localized in momentum space (centered on $\vec{p}_A$ and $\vec{p}_B$), all factors that are smooth functions of $\vec{k}_i$ can be evaluated at $\vec{p}_i$ and pulled out of the integrals. Show that this yields

$$d\sigma = \left(\prod_f \frac{d^3p_f}{(2\pi)^3} \frac{1}{2E_f}\right) \frac{|\mathcal{M}(p_A, p_B \to \{p_f\})|^2}{2E_A 2E_B |v_A - v_B|} \int \frac{d^3k_A}{(2\pi)^3} \int \frac{d^3k_B}{(2\pi)^3}$$
$$\cdot \, |\varphi_A(\vec{k}_A)|^2 |\varphi_B(\vec{k}_B)|^2 (2\pi)^4 \delta^{(4)}(k_A + k_B - \sum_f p_f) \, ,$$

where $|v_A - v_B|$ is the relative velocity of the beams as viewed from the laboratory frame.

d) Real detectors have finite resolution and therefore sum incoherently over momentum bins of finite size. The bin size is normally larger than the momentum spread of the initial wave packets. This allows us to approximate $k_A + k_B$ in the δ-function by its central value, $p_A + p_B$. Show that this finally yields

$$d\sigma = \left(\prod_f \frac{d^3p_f}{(2\pi)^3} \frac{1}{2E_f}\right) \frac{|\mathcal{M}(p_A, p_B \to \{p_f\})|^2}{2E_A 2E_B |v_A - v_B|} (2\pi)^4 \delta^{(4)}(p_A + p_B - \sum_f p_f)$$

as stated in class. Note that all dependence on the shape of the wave packets has disappeared.

e) Now consider the special case of two particles in the final state. Use the δ-function to integrate over 4 of the 6 momentum components (in spherical coordinates) to show that

$$\left(\frac{d\sigma}{d\Omega}\right)_{\text{CM}} = \frac{|\mathcal{M}(p_A, p_B \to p_1, p_2)|^2}{2E_A 2E_B |v_A - v_B|} \frac{|\vec{p}_1|}{(2\pi)^2 4E_{\text{CM}}} \, ,$$

where E_{CM} is the total initial energy. Show how to compute $|\vec{p}_1|$ from E_{CM}, m_1, and m_2.

f) Show that if all four particles have identical masses, this becomes

$$\left(\frac{d\sigma}{d\Omega}\right)_{\text{CM}} = \frac{|\mathcal{M}|^2}{64\pi^2 E_{\text{CM}}^2} \, .$$

Problem 2 Decays of scalar particles

a) Consider a theory of two real scalar fields Φ and φ with masses M and m, respectively, and an interaction $\mathcal{L}_{\text{int}} = g\Phi\varphi\varphi$. The interaction term allows a Φ particle to decay into two φ particles, provided that $M > 2m$. Assume that this condition is satisfied and calculate the lifetime of the Φ particle to lowest order in g.

b) Now consider a theory of a real scalar field φ and a complex scalar field χ with $\mathcal{L}_{\text{int}} = g\varphi\chi^\dagger\chi$. Assuming that $m_\varphi > 2m_\chi$, calculate the lifetime of the φ particle to lowest order in g.

Hint 1: It is probably easiest to use "contractions with external legs" (see Sec. 5.8.3 of the lecture) to show that $\mathcal{M} = 2g$ in a) and $\mathcal{M} = g$ in b).

Hint 2: Pay attention to the difference between distinguishable and indistinguishable particles.