Universität Regensburg, Institut für Theoretische Physik
Prof. Dr. Tilo Wettig SS 2024

Quantum Electrodynamics

Problem Set 9 (for the exercises on June 18 and 19)

Problem 1 Chiral transformations

A chiral transformation of a spinor is defined by

$$\psi \to e^{i\alpha\gamma^5}\psi \, . \tag{1}$$

a) How does the conjugate spinor $\bar{\psi}$ transform under (1)? Recall $\{\gamma^\mu, \gamma^5\} = 0$.

b) How does the object $V^\mu = \bar{\psi}\gamma^\mu\psi$ transform under (1)?

c) Show that the Dirac Lagrangian $\mathcal{L} = \bar{\psi}(i\gamma^\mu\partial_\mu - m)\psi$ is invariant under (1) for $m = 0$ and that it is not invariant if $m \neq 0$.

d) Show that for $m = 0$ the Noether current is given by $j_5^\mu = \bar{\psi}\gamma^\mu\gamma^5\psi$ and that for $m \neq 0$ it satisfies $\partial_\mu j_5^\mu = 2im\bar{\psi}\gamma^5\psi$.

Problem 2 Fierz transformations

In the lecture we have discussed Dirac field bilinears such as $\bar{\psi}\gamma^\mu\psi$. They can be generalized to objects of the form

$$\bar{u}_1\Gamma^A u_2 \, ,$$

where the u_i are 4-component Dirac spinors and Γ^A is any of the following 16 (= 1+4+6+4+1) combinations of Dirac matrices,

$$\mathbb{1} \, , \quad \gamma^\mu \, , \quad \sigma^{\mu\nu} = \tfrac{i}{2}[\gamma^\mu, \gamma^\nu] \quad (\mu < \nu) \, , \quad \gamma^\mu\gamma^5 \, , \quad \gamma^5 \, .$$

For practical calculations it is sometimes helpful to use rearrangement formulas for products of such bilinears. The general Fierz identity is given by the equation

$$\left(\bar{u}_1\Gamma^A u_2\right)\left(\bar{u}_3\Gamma^B u_4\right) = \sum_{C,D} C^{AB}_{CD}\left(\bar{u}_1\Gamma^C u_4\right)\left(\bar{u}_3\Gamma^D u_2\right)$$

with coefficients C^{AB}_{CD} to be determined.

a) Note that the Γ^A form a complete orthogonal basis for the space of 4×4 matrices, where orthogonality is defined with respect to a scalar product given by the trace. Normalize the 16 matrices Γ^A to the convention

$$\mathrm{tr}(\Gamma^A\Gamma^B) = 4\delta^{AB} \, .$$

This gives $\Gamma^A = \{\mathbb{1}, \gamma^0, i\gamma^i, \ldots\}$. Write down all 16 elements of this set.

b) Using the completeness of the 16 Γ^A matrices, show that

$$C_{CD}^{AB} = \frac{1}{16}\,\mathrm{tr}(\Gamma^C\Gamma^A\Gamma^D\Gamma^B)\,.$$

Hint: The completeness implies that any 4×4 matrix can be written as a linear combination of the Γ^A. In particular, we can write

$$u_i\bar{u}_j = \sum_A c_A^{ij}\Gamma^A$$

with some coefficients c_A^{ij}. Use the cyclic property of the trace to show that $c_A^{ij} = \frac{1}{4}(\bar{u}_j\Gamma^A u_i)$. This result can be used to derive C_{CD}^{AB}.

c) Work out explicitly the Fierz transformation laws for the products $(\bar{u}_1u_2)(\bar{u}_3u_4)$ and $(\bar{u}_1\gamma^\mu u_2)(\bar{u}_3\gamma_\mu u_4)$.