Universität Regensburg, Institut für Theoretische Physik SS 2024
Prof. Dr. Tilo Wettig

Quantum Electrodynamics
Problem Set 7 (for the exercises on June 04 and 05)

Problem 1 Gordon identity

Derive the Gordon identity,

$$\bar{u}(p')\gamma^\mu u(p) = \bar{u}(p')\left[\frac{p'^\mu + p^\mu}{2m} + \frac{i\sigma^{\mu\nu}q_\nu}{2m}\right]u(p)\,,$$

where $q = p' - p$, $\sigma^{\mu\nu} = \frac{i}{2}[\gamma^\mu, \gamma^\nu]$, and $u(p)$ satisfies $(\not{p} - m)u(p) = 0$. This identity will be useful later in the course when we discuss the electron vertex function.

Problem 2 The time-evolution operator of the Klein-Gordon field

Let $U(t, t')$ with $t \geq t'$ be an operator that satisfies the differential equation

$$\frac{\partial U(t,t')}{\partial t} = -iH_I(t)U(t,t')$$

with initial condition $U(t,t) = \mathbb{1}$. Note that $H_I(t)$ is an operator.

a) Show (by differentiating with respect to t) that

$$\begin{aligned}U(t,t') = \mathbb{1} &+ (-i)\int_{t'}^{t} dt_1\, H_I(t_1) + (-i)^2\int_{t'}^{t} dt_1 \int_{t'}^{t_1} dt_2\, H_I(t_1)H_I(t_2) \\ &+ (-i)^3 \int_{t'}^{t} dt_1 \int_{t'}^{t_1} dt_2 \int_{t'}^{t_2} dt_3\, H_I(t_1)H_I(t_2)H_I(t_3) + \ldots\end{aligned}$$

is a solution of the differential equation.

b) Show that the terms in the above solution can be rewritten in the form

$$\int_{t'}^{t} dt_1 \cdots \int_{t'}^{t_{n-1}} dt_n\, H_I(t_1)\cdots H_I(t_n) = \frac{1}{n!}\int_{t'}^{t} dt_1 \cdots dt_n\, T\{H_I(t_1)\cdots H_I(t_n)\}\,.$$

This proves the expression for U stated in class:

$$U(t,t') = \sum_{n=0}^{\infty} \frac{(-i)^n}{n!}\int_{t'}^{t} dt_1 \cdots dt_n\, T\{H_I(t_1)\cdots H_I(t_n)\} = T\left\{e^{-i\int_{t'}^{t} dt''\, H_I(t'')}\right\}.$$

c) Show that

$$U(t,t') = e^{iH_0(t-t_0)}e^{-iH(t-t')}e^{-iH_0(t'-t_0)}\,,$$

where H_0 and $H = H_0 + H_{\text{int}}$ are operators in the Schrödinger picture, satisfies the differential equation and the initial condition. Here, t_0 is the reference time at which the Heisenberg and interaction pictures coincide. This expression makes it obvious that U is unitary.

d) Show that for $t_1 \geq t_2 \geq t_3$ we have

$$\begin{aligned}U(t_1,t_2)U(t_2,t_3) &= U(t_1,t_3)\,,\\ U(t_1,t_3)U^\dagger(t_2,t_3) &= U(t_1,t_2)\,.\end{aligned}$$

Problem 3 Proof of Wick's theorem

Prove Wick's theorem,

$$T\{\varphi(x_1)\cdots\varphi(x_n)\} = N\{\varphi(x_1)\cdots\varphi(x_n) + \text{all possible contractions}\}\,,$$

where T and N denote time-ordering and normal-ordering, respectively.
Hint: Use induction, starting with the case of $n = 2$ that was already proven in class. It suffices to consider the case $x_1^0 \geq \cdots \geq x_n^0$ (why?).