Universität Regensburg, Institut für Theoretische Physik SS 2024
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Quantum Electrodynamics
Problem Set 6 (for the exercises on May 28 and 29)

Problem 1 Time reversal

a) We have seen in class that T reverses three-momentum and spin, so we need to learn how to deal with spin flips. Let ξ^s ($s = 1, 2$) be a two-component spinor. We claim that the flipped spinor is given by

$$\xi^{-s} := -i\sigma^2(\xi^s)^* .$$

To prove this, show that $\vec{\sigma}\sigma^2 = \sigma^2(-\vec{\sigma}^*)$ and use this identity to show that, if a two-component spinor ξ satisfies $(\hat{n}\cdot\vec{\sigma})\xi = +\xi$ for some axis $\hat{n}$, then $(\hat{n}\cdot\vec{\sigma})(-i\sigma^2\xi^*) = -(-i\sigma^2\xi^*)$.

b) To deal with momentum reversal, define $\tilde{p} = (p^0, -\vec{p})$ and show that

$$\sqrt{\tilde{p}\cdot\sigma}\,\sigma^2 = \sigma^2\sqrt{p\cdot\sigma^*}\,, \qquad \sqrt{\tilde{p}\cdot\bar{\sigma}}\,\sigma^2 = \sigma^2\sqrt{p\cdot\bar{\sigma}^*}\,.$$

Hint: $\sqrt{p\cdot\sigma}$ is a 2×2 matrix, so it can be written as a linear combination of $\mathbb{1}_2$ and the three Pauli matrices, $\sqrt{p\cdot\sigma} = \alpha_0\mathbb{1}_2 + \alpha_i\sigma^i$. The coefficients α_μ were given in Sec. 3.3 of the lecture.

c) In Problem 5.1 explicit expressions for u and v were given,

$$u^s(p) = \begin{pmatrix}\sqrt{p\cdot\sigma}\,\xi^s\\ \sqrt{p\cdot\bar{\sigma}}\,\xi^s\end{pmatrix} \qquad \text{and} \qquad v^s(p) = \begin{pmatrix}\sqrt{p\cdot\sigma}\,\eta^s\\ -\sqrt{p\cdot\bar{\sigma}}\,\eta^s\end{pmatrix} \tag{1}$$

with $s = 1, 2$, $\xi^{r\dagger}\xi^s = \delta^{rs}$ and $\eta^{r\dagger}\eta^s = \delta^{rs}$. Using these expressions, show that the Dirac spinors with reversed momentum and spin are given by

$$u^{-s}(\tilde{p}) = -\gamma^1\gamma^3[u^s(p)]^*\,, \qquad v^{-s}(\tilde{p}) = -\gamma^1\gamma^3[v^s(p)]^*\,.$$

d) If in part a) we choose $\xi^1 = \begin{pmatrix}1\\0\end{pmatrix}$ and $\xi^2 = \begin{pmatrix}0\\1\end{pmatrix}$, then $\xi^{-1} = \xi^2$ and $\xi^{-2} = -\xi^1$ (this does not result in any loss of generality since the axis $\hat{n}$ is arbitrary). In analogy, we define

$$a_{\vec{p}}^{-1} = a_{\vec{p}}^2\,, \qquad a_{\vec{p}}^{-2} = -a_{\vec{p}}^1$$

and similarly for $b_{\vec{p}}^s$. The action of T on the fermion annihilation operators is defined by

$$Ta_{\vec{p}}^sT^{-1} = a_{-\vec{p}}^{-s}\,, \qquad Tb_{\vec{p}}^sT^{-1} = b_{-\vec{p}}^{-s}\,,$$

i.e., momentum and spin are reversed as required. Show that the Dirac field and the various field bilinears transform under T as follows,

$$\begin{aligned}
T\psi(t,\vec{x})T^{-1} &= \gamma^1\gamma^3\psi(-t,\vec{x})\\
T\bar{\psi}(t,\vec{x})T^{-1} &= \bar{\psi}(-t,\vec{x})(-\gamma^1\gamma^3)\\
T\bar{\psi}\psi T^{-1} &\to \bar{\psi}\psi\\
Ti\bar{\psi}\gamma^5\psi T^{-1} &\to -i\bar{\psi}\gamma^5\psi
\end{aligned}$$

$$T\bar\psi\gamma^\mu\psi T^{-1} \to \begin{cases} +\bar\psi\gamma^\mu\psi & \text{for } \mu = 0 \\ -\bar\psi\gamma^\mu\psi & \text{for } \mu = 1,2,3 \end{cases}$$
$$T\bar\psi\gamma^\mu\gamma^5\psi T^{-1} \to \begin{cases} +\bar\psi\gamma^\mu\gamma^5\psi & \text{for } \mu = 0 \\ -\bar\psi\gamma^\mu\gamma^5\psi & \text{for } \mu = 1,2,3 \end{cases}$$
$$Ti\bar\psi[\gamma^\mu,\gamma^\nu]\psi T^{-1} \to \begin{cases} -i\bar\psi[\gamma^\mu,\gamma^\nu]\psi & \text{for } \mu\nu = ij \\ +i\bar\psi[\gamma^\mu,\gamma^\nu]\psi & \text{for } \mu\nu = 0i \text{ or } i0 \end{cases}$$

Hint: Recall that T is an anti-unitary operator that involves complex conjugation.

Problem 2 Charge conjugation

In Eq. (1) the choice of ξ and η is arbitrary, subject to the constraints $\xi^{r\dagger}\xi^s = \delta^{rs}$ and $\eta^{r\dagger}\eta^s = \delta^{rs}$. You have shown in Problem 4.4 that for antifermions the spin is reversed with respect to the fermions. Therefore it is sensible to choose $\eta^s = \xi^{-s}$ with ξ^{-s} defined in Problem 1a), i.e.,

$$v^s(p) = \begin{pmatrix} \sqrt{p\cdot\sigma}\,\xi^{-s} \\ -\sqrt{p\cdot\bar\sigma}\,\xi^{-s} \end{pmatrix}.$$

a) Using this convention and the results of Problem 1b), show that

$$u^s(p) = -i\gamma^2[v^s(p)]^*, \qquad v^s(p) = -i\gamma^2[u^s(p)]^*.$$

b) Using

$$Ca^s_{\vec p}C^{-1} = b^s_{\vec p}, \qquad Cb^s_{\vec p}C^{-1} = a^s_{\vec p},$$

show that the Dirac field and the various field bilinears transform under C as follows,

$$\begin{aligned}
C\psi C^{-1} &= -i\gamma^2\psi^* = -i(\bar\psi\gamma^0\gamma^2)^T \\
C\bar\psi C^{-1} &= -i(\gamma^0\gamma^2\psi)^T \\
C\bar\psi\psi C^{-1} &= \bar\psi\psi \\
Ci\bar\psi\gamma^5\psi C^{-1} &= i\bar\psi\gamma^5\psi \\
C\bar\psi\gamma^\mu\psi C^{-1} &= -\bar\psi\gamma^\mu\psi \\
C\bar\psi\gamma^\mu\gamma^5\psi C^{-1} &= \bar\psi\gamma^\mu\gamma^5\psi \\
Ci\bar\psi[\gamma^\mu,\gamma^\nu]\psi C^{-1} &= -i\bar\psi[\gamma^\mu,\gamma^\nu]\psi
\end{aligned}$$

Problem 3 The free electromagnetic field

a) Derive the Hamiltonian of the free electromagnetic field,

$$H = \int d^3x[\pi^\mu(x)\dot A_\mu(x) - \mathcal{L}] = \int \frac{d^3p}{(2\pi)^3}\sum_{r=0}^{3}\zeta_r E_{\vec p}\, a^{r\dagger}_{\vec p} a^r_{\vec p},$$

using $\mathcal{L} = -\frac{1}{2}(\partial_\mu A_\nu)(\partial^\mu A^\nu)$. You can either use the expansion of the field in terms of ladder operators or find a shortcut by comparing the Lagrangian to that of the Klein-Gordon field.

b) Using the special choice for the polarization vectors $\varepsilon_r(p)$ given at the end of Sec. 4.1 of the lecture, show that the Gupta-Bleuler condition, $\partial_\mu A^{\mu+}(x)|\Psi\rangle = 0$, leads to the condition

$$(a^3_{\vec p} - a^0_{\vec p})|\Psi\rangle = 0 \quad \text{for all } \vec p.$$

c) Using the expression for H from part a), show that the condition you derived in part b) implies that only transverse photons contribute to the energy, i.e.,

$$\langle\Psi|H|\Psi\rangle = \langle\Psi|\int \frac{d^3p}{(2\pi)^3}\sum_{r=1}^{2} E_{\vec p}\, a^{r\dagger}_{\vec p} a^r_{\vec p}|\Psi\rangle.$$