Universität Regensburg, Institut für Theoretische Physik SS 2024
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Quantum Electrodynamics
Problem Set 5 (for the exercises on May 22)

There is no teaching on May 21 because of the holiday weekend. If you are in the Tuesday group, please try to attend the Wednesday group.

Problem 1 Solutions of the free Dirac equation

We have seen in class that the free Dirac equation, $(i\partial\!\!\!/-m)\psi(x)=0$, has solutions with positive and negative frequency,

$$\psi(x)=u(p)e^{-ip\cdot x} \quad \text{and} \quad \psi(x)=v(p)e^{+ip\cdot x}$$

with $p^2=m^2$ and $p^0>0$. The Dirac spinors u and v have the form

$$u^s(p)=\begin{pmatrix}\sqrt{p\cdot\sigma}\,\xi^s\\ \sqrt{p\cdot\bar\sigma}\,\xi^s\end{pmatrix}, \qquad v^s(p)=\begin{pmatrix}\sqrt{p\cdot\sigma}\,\eta^s\\ -\sqrt{p\cdot\bar\sigma}\,\eta^s\end{pmatrix}$$

with $s=1,2$. The two-spinors ξ and η are normalized by $\xi^{r\dagger}\xi^s=\delta^{rs}$ and $\eta^{r\dagger}\eta^s=\delta^{rs}$.

a) Show that $(p\cdot\sigma)(p\cdot\bar\sigma)=p^2=m^2$, where $\sigma^\mu=(\mathbb{1}_2,\vec\sigma)$ and $\bar\sigma^\mu=(\mathbb{1}_2,-\vec\sigma)$.

b) Show that the Dirac spinors must satisfy the equations $(p\!\!\!/-m)u(p)=0$ and $(p\!\!\!/+m)v(p)=0$ and verify that these equations are indeed satisfied by $u^s(p)$ and $v^s(p)$ given above.

c) Show that

$$\begin{aligned}
\bar u^r(p)u^s(p)&=2m\delta^{rs}\,, & u^{r\dagger}(p)u^s(p)&=2E_{\vec p}\delta^{rs}\,,\\
\bar v^r(p)v^s(p)&=-2m\delta^{rs}\,, & v^{r\dagger}(p)v^s(p)&=2E_{\vec p}\delta^{rs}\,,\\
\bar u^r(p)v^s(p)&=\bar v^r(p)u^s(p)=0\,, & u^{r\dagger}(\vec p)v^s(-\vec p)&=v^{r\dagger}(\vec p)u^s(-\vec p)=0\,.
\end{aligned}$$

d) Show that

$$\sum_s u^s(p)\bar u^s(p)=p\!\!\!/+m \qquad \text{and} \qquad \sum_s v^s(p)\bar v^s(p)=p\!\!\!/-m\,.$$

Problem 2 Effect of continuous Lorentz transformations on the Dirac field

In this problem we consider continuous (i.e., proper and orthochronous) Lorentz transformations. Let $U(\Lambda)$ be the operator that implements such a Lorentz transformation Λ on the states of the Hilbert space, i.e., $|\text{state}\rangle\to U(\Lambda)|\text{state}\rangle$.

a) Consider one-particle states of the form

$$|\vec p,s\rangle=\sqrt{2E_{\vec p}}\,a^{s\dagger}_{\vec p}|0\rangle\,.$$

We have seen in class that the inner product of two such states is Lorentz invariant. Use this fact to show that $U(\Lambda)$ is unitary.

b) Assume that the boost or rotation axis is parallel to the spin quantization axis and show that the above equation implies that $a^s_{\vec{p}}$ transforms like

$$U(\Lambda)\left[\sqrt{E_{\vec{p}}}\,a^s_{\vec{p}}\right]U(\Lambda^{-1}) = \sqrt{E_{\Lambda\vec{p}}}\,a^s_{\Lambda\vec{p}}\,.$$

Hint: Consider a two-particle state to motivate the $U(\Lambda^{-1})$ on the right of $a^s_{\vec{p}}$.

c) We already know that the field $\psi(x)$ should transform under continuous Lorentz transformations like

$$\psi(x) \to \psi'(x) = \Lambda_{\frac{1}{2}}\psi(\Lambda^{-1}x)\,. \tag{1}$$

Using

$$\psi(x) = \int \frac{d^3p}{(2\pi)^3}\frac{1}{\sqrt{2E_{\vec{p}}}}\sum_s \left(a^s_{\vec{p}}u^s(p)e^{-ip\cdot x} + b^{s\dagger}_{\vec{p}}v^s(p)e^{ip\cdot x}\right),$$

show that $U(\Lambda)$ implements the correct transformation on the field operator $\psi(x)$, i.e.,

$$U(\Lambda)\psi(x)U(\Lambda^{-1}) = \Lambda^{-1}_{\frac{1}{2}}\psi(\Lambda x)\,. \tag{2}$$

Hint: Relate $u^s(p)$ to $u^s(\Lambda p)$ and $v^s(p)$ to $v^s(\Lambda p)$ using the known transformation properties of Dirac spinors.

d) Explain the difference between the right-hand sides of Eqs. (1) and (2).

Problem 3 The quantized Dirac field

Verify the following results for the free Dirac field that were stated but not proven in class. Use the expression for $\psi(x)$ given in Problem 2c) and its Hermitian conjugate.

a) The momentum operator is

$$\vec{P} = \int d^3x\,\psi^\dagger(-i\vec{\nabla})\psi = \int \frac{d^3p}{(2\pi)^3}\sum_s \vec{p}(a^{s\dagger}_{\vec{p}}a^s_{\vec{p}} + b^{s\dagger}_{\vec{p}}b^s_{\vec{p}})\,.$$

b) The invariance of $\mathcal{L}$ under a global change of phase, $\psi(x) \to e^{i\alpha}\psi(x)$, yields the conserved Noether current

$$j^\mu = \bar{\psi}\gamma^\mu\psi$$

and the conserved charge

$$Q = \int d^3x\,j^0 = \int d^3x\,\psi^\dagger\psi = \int \frac{d^3p}{(2\pi)^3}\sum_s (a^{s\dagger}_{\vec{p}}a^s_{\vec{p}} - b^{s\dagger}_{\vec{p}}b^s_{\vec{p}})\,.$$

c) The Hamilton operator is

$$H = \int d^3x\,\bar{\psi}(-i\vec{\gamma}\cdot\vec{\nabla} + m)\psi = \int \frac{d^3p}{(2\pi)^3}\sum_s E_{\vec{p}}(a^{s\dagger}_{\vec{p}}a^s_{\vec{p}} + b^{s\dagger}_{\vec{p}}b^s_{\vec{p}})\,.$$

Hint: Use $(\not{p} - m)u = 0$, $(\not{p} + m)v = 0$, and other results of Problem 1.