Universität Regensburg, Institut für Theoretische Physik
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# Quantum Electrodynamics
## Problem Set 4 (for the exercises on May 14 and 15)

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### Problem 1 Spinor representation of the Lorentz algebra

Assume that we have four $n \times n$ matrices $\gamma^\mu$ satisfying the anticommutation relations

$$\{\gamma^\mu, \gamma^\nu\} \equiv \gamma^\mu \gamma^\nu + \gamma^\nu \gamma^\mu = 2g^{\mu\nu} \mathbb{1}_n \,.$$

We only know these relations but not the explicit form of the $\gamma^\mu$. Show that the $n \times n$ matrices

$$S^{\mu\nu} = \frac{i}{4}[\gamma^\mu, \gamma^\nu]$$

satisfy the commutation relations of the Lorentz algebra, i.e.,

$$[S^{\mu\nu}, S^{\rho\sigma}] = i(g^{\nu\rho} S^{\mu\sigma} - g^{\mu\rho} S^{\nu\sigma} - g^{\nu\sigma} S^{\mu\rho} + g^{\mu\sigma} S^{\nu\rho}) \,.$$

Hint: First prove $[S^{\mu\nu}, \gamma^\rho] = i\,(\gamma^\mu g^{\nu\rho} - \gamma^\nu g^{\mu\rho})$.

### Problem 2 A property of the $\gamma$-matrices

Show that the $\gamma$-matrices satisfy the relation

$$\Lambda_{\frac{1}{2}}^{-1} \gamma^\mu \Lambda_{\frac{1}{2}} = \Lambda^\mu{}_\nu \gamma^\nu \,,$$

where

$$\Lambda_{\frac{1}{2}} = \exp\left(-\frac{i}{2}\omega_{\mu\nu} S^{\mu\nu}\right) \quad \text{with} \quad S^{\mu\nu} = \frac{i}{4}[\gamma^\mu, \gamma^\nu] \,,$$
$$\Lambda = \exp\left(-\frac{i}{2}\omega_{\mu\nu} \mathcal{J}^{\mu\nu}\right) \quad \text{with} \quad (\mathcal{J}^{\mu\nu})_{\alpha\beta} = i(\delta^\mu_\alpha \delta^\nu_\beta - \delta^\mu_\beta \delta^\nu_\alpha) \,.$$

Hint: Substitute $\omega_{\mu\nu} \to \alpha\omega_{\mu\nu}$ and show that the matrices $\gamma^\mu(\alpha) = \Lambda^{-1}_{1/2}(\alpha)\gamma^\mu \Lambda_{1/2}(\alpha)$ and $\tilde{\gamma}^\mu(\alpha) = \Lambda^\mu{}_\nu(\alpha)\gamma^\nu$ satisfy the same differential equation, i.e., $\partial_\alpha \gamma^\mu(\alpha) = \partial_\alpha \tilde{\gamma}^\mu(\alpha)$. The result from the hint in Problem 1 may also be useful.

### Problem 3 Transformation properties of $\bar{\psi}\psi$ and $\bar{\psi}\gamma^\mu\psi$

a) We know that $\psi$ transforms like $\psi \to \Lambda_{\frac{1}{2}}\psi$. Show that $\bar{\psi} \equiv \psi^\dagger \gamma^0$ transforms like $\bar{\psi} \to \bar{\psi}\Lambda_{\frac{1}{2}}^{-1}$. Therefore $\bar{\psi}\psi$ is a Lorentz scalar.

   Hint: Show that $(S^{\mu\nu})^\dagger \gamma^0 = \gamma^0 S^{\mu\nu}$.

b) Show that $\bar{\psi}\gamma^\mu\psi$ is a Lorentz vector.

   Hint: Use the property of the $\gamma$-matrices you proved in Problem 2.

**Problem 4 Orbital angular momentum and spin**

In this problem, we will construct the angular momentum operator and show that the particles created by $a^{s\dagger}_{\vec{p}}$ and $b^{s\dagger}_{\vec{p}}$ have spin $\frac{1}{2}$.

a) Since the Dirac Lagrangian is invariant under Lorentz transformations, it is invariant under rotations. Consider an infinitesimal rotation by an angle $\theta$ about the $z$-axis, i.e., $\omega_{12} = -\omega_{21} = \theta$, so that

$$\Lambda_{\frac{1}{2}} \approx \mathbb{1} - \frac{i}{2}\theta\Sigma^3 \,,$$

where $\Sigma^3 = \mathrm{diag}(\sigma^3, \sigma^3)$. Compute the change

$$\delta\psi = \psi'(x) - \psi(x) = \Lambda_{\frac{1}{2}}\psi(\Lambda^{-1}x) - \psi(x)$$

in the field at the point $x$ and show that the time-component of the conserved Noether current is

$$j^0 = -i\bar{\psi}\gamma^0(x\partial_y - y\partial_x + \frac{i}{2}\Sigma^3)\psi \,.$$

Doing similar calculations for rotations about the $x$- and $y$-axes, one finds the angular momentum operator

$$\vec{J} = \int d^3x\, \psi^\dagger \left\{ \vec{x} \times (-i\vec{\nabla}) + \frac{1}{2}\vec{\Sigma} \right\} \psi \,.$$

b) Express $\psi$ and $\psi^\dagger$ in terms of ladder operators to show that, at $t = 0$,

$$J_z = \int d^3x \int \frac{d^3p}{(2\pi)^3}\frac{d^3p'}{(2\pi)^3}\frac{1}{\sqrt{2E_{\vec{p}}2E_{\vec{p}'}}}e^{i(\vec{p}-\vec{p}')\cdot\vec{x}}$$
$$\times \sum_{rr'} \left( a^{r'\dagger}_{\vec{p}'}u^{r'\dagger}(\vec{p}') + b^{r'}_{-\vec{p}'}v^{r'\dagger}(-\vec{p}') \right) \left( xp^2 - yp^1 + \frac{1}{2}\Sigma^3 \right) \left( a^r_{\vec{p}}u^r(\vec{p}) + b^{r\dagger}_{-\vec{p}}v^r(-\vec{p}) \right) .$$

In the following we consider particles at rest. We want to show that $a^{s\dagger}_0|0\rangle$ is an eigenstate of $J_z$ with eigenvalue $\pm\frac{1}{2}$. Since $J_z$ must annihilate the vacuum, we have $J_z a^{s\dagger}_0|0\rangle = [J_z, a^{s\dagger}_0]|0\rangle$. Show that the only nonzero term in the latter expression leads to

$$J_z a^{s\dagger}_0|0\rangle = \frac{1}{2m}\sum_r \left( u^{r\dagger}(0)\frac{\Sigma^3}{2}u^s(0) \right) a^{r\dagger}_0|0\rangle = \sum_r \left( \xi^{r\dagger}\frac{\sigma^3}{2}\xi^s \right) a^{r\dagger}_0|0\rangle \,,$$

where $u(0)$ means $u(p = (m, \vec{0}))$. Choose the spinors $\xi^r$ to be eigenstates of $\sigma^3$ to show that we indeed have

$$J_z a^{s\dagger}_0|0\rangle = \pm\frac{1}{2}a^{s\dagger}_0|0\rangle \,.$$

c) Show that for antifermions the sign is reversed, i.e.,

$$J_z b^{s\dagger}_0|0\rangle = \mp\frac{1}{2}b^{s\dagger}_0|0\rangle \,.$$

d) (optional) To justify the trick used in part b), show explicitly that $J_z$ annihilates the vacuum.