Universität Regensburg, Institut für Theoretische Physik SS 2024
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Quantum Electrodynamics
Problem Set 3 (for the exercises on May 7 and 8)

Problem 1 The complex scalar field

Consider the field theory of a complex-valued scalar field obeying the Klein-Gordon equation. The action of this theory is

$$S = \int d^4x \, (\partial_\mu \varphi^* \partial^\mu \varphi - m^2 \varphi^* \varphi) \, .$$

It is easiest to analyze this theory by considering $\varphi(x)$ and $\varphi^*(x)$, rather than the real and imaginary parts of $\varphi(x)$, as the basic dynamical variables.

a) Find the momenta conjugate to $\varphi(x)$ and $\varphi^*(x)$. Quantize the fields by postulating canonical commutation relations. Show that the Hamiltonian is

$$H = \int d^3x \, (\pi^\dagger \pi + \vec{\nabla}\varphi^\dagger \cdot \vec{\nabla}\varphi + m^2 \varphi^\dagger \varphi) \, .$$

Compute the Heisenberg equation of motion for $\varphi(x)$ and show that it is indeed the Klein-Gordon equation.

b) Diagonalize H by introducing creation and annihilation operators. Show that the theory contains two sets of particles of mass m.

Hint: Since the operator φ is no longer Hermitian, its Fourier transform contains two independent operators $a_{\vec{p}}$ and $b^\dagger_{\vec{p}}$ (instead of $a_{\vec{p}}$ and $a^\dagger_{\vec{p}}$ in the Hermitian case). It may be convenient to introduce Hermitian field operators $(\varphi + \varphi^\dagger)/\sqrt{2}$ and $(\varphi - \varphi^\dagger)/\sqrt{2}i$, and similarly for π and $\pi^\dagger$.

c) Show that the Lagrangian is invariant under a global change of the phase of φ (i.e., $\varphi(x) \to e^{i\alpha}\varphi(x)$) and that the conserved charge corresponding to this symmetry transformation is

$$Q = i \int d^3x \, (\varphi^\dagger \pi^\dagger - \varphi \pi) \, .$$

Rewrite Q in terms of creation and annihilation operators and evaluate the charge of the particles of each type.

Problem 2 Green's functions of the Klein-Gordon operator

a) Show that the function

$$D_R(x-y) = \theta(x^0 - y^0)\langle 0|[\varphi(x), \varphi(y)]|0\rangle$$

satisfies the equation

$$(\partial^2 + m^2) D_R(x-y) = -i\delta^{(4)}(x-y)$$

and is therefore a (retarded) Green's function of the Klein-Gordon operator $\partial^2 + m^2$.

b) Show that the Feynman propagator

$$D_F(x-y) = \langle 0|T\varphi(x)\varphi(y)|0\rangle$$

is also a Green's function of the Klein-Gordon operator.

Problem 3 Particle creation by a classical source

Consider a real Klein-Gordon field coupled to an external, classical source j. The Lagrangian is given by

$$\mathcal{L} = \frac{1}{2}\partial_\mu\varphi\,\partial^\mu\varphi - \frac{1}{2}m^2\varphi^2 + j(x)\varphi(x)\,,$$

where $j(x)$ is a known real function that is nonzero only for a finite time interval.

a) Show that the equation of motion is

$$(\partial^2 + m^2)\varphi(x) = j(x)\,.$$

b) Before $j(x)$ is turned on, $\varphi(x)$ has the form

$$\varphi_0(x) = \int \frac{d^3p}{(2\pi)^3}\,\frac{1}{\sqrt{2E_{\vec{p}}}}\left(a_{\vec{p}}\,e^{-ip\cdot x} + a^\dagger_{\vec{p}}\,e^{ip\cdot x}\right)\Big|_{p^0=E_{\vec{p}}}\,.$$

With a source, we can construct $\varphi(x)$ using the retarded Green's function:

$$\varphi(x) = \varphi_0(x) + i\int d^4y\,D_R(x-y)j(y)\,.$$

Show that after j has acted (i.e., for times large enough so that j is zero again), we have

$$\varphi(x) = \int \frac{d^3p}{(2\pi)^3}\,\frac{1}{\sqrt{2E_{\vec{p}}}}\left\{\left(a_{\vec{p}} + \frac{i}{\sqrt{2E_{\vec{p}}}}\tilde{j}(p)\right)e^{-ip\cdot x} + \text{h.c.}\right\},$$

where

$$\tilde{j}(p) = \int d^4y\,e^{ip\cdot y}j(y)$$

is the Fourier transform of j for 4-momenta p such that $p^2 = m^2$.

c) Show that after j has acted, the Hamiltonian is given by

$$H = \int \frac{d^3p}{(2\pi)^3}\,E_{\vec{p}}\left(a^\dagger_{\vec{p}} - \frac{i}{\sqrt{2E_{\vec{p}}}}\tilde{j}^*(p)\right)\left(a_{\vec{p}} + \frac{i}{\sqrt{2E_{\vec{p}}}}\tilde{j}(p)\right)$$

and that the energy of the system after the source has been turned off is

$$\langle 0|H|0\rangle = \int \frac{d^3p}{(2\pi)^3}\,\frac{1}{2}|\tilde{j}(p)|^2 = \int \frac{d^3p}{(2\pi)^3}\,E_{\vec{p}}\,\frac{|\tilde{j}(p)|^2}{2E_{\vec{p}}}\,,$$

where $|0\rangle$ is the ground state of the free theory. These results show that $|\tilde{j}(p)|^2/2E_{\vec{p}}$ is the probability density for creating a particle in the mode p. The total number of particles created is then

$$N = \int dN = \int \frac{d^3p}{(2\pi)^3}\,\frac{|\tilde{j}(p)|^2}{2E_{\vec{p}}}\,.$$

Note that particles are created only by those Fourier components of j that satisfy the on-mass-shell condition $p^2 = m^2$.