Universität Regensburg, Institut für Theoretische Physik SS 2024
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Quantum Electrodynamics
Problem Set 2 (for the exercise on April 30)

May 1 is a holiday. If you are in the Wednesday group, please try to attend the Tuesday group.

Problem 1 Fourier components of the Klein-Gordon field and ladder operators

As discussed in the lecture, we treat the Fourier components of the Klein-Gordon field as independent oscillators,

$$\varphi(\vec{x}) = \int \frac{d^3p}{(2\pi)^3} \frac{1}{\sqrt{2E_{\vec{p}}}} \Big(a_{\vec{p}} + a^\dagger_{-\vec{p}} \Big) e^{i\vec{p}\cdot\vec{x}} \,,$$

$$\pi(\vec{x}) = \int \frac{d^3p}{(2\pi)^3} (-i) \sqrt{\frac{E_{\vec{p}}}{2}} \Big(a_{\vec{p}} - a^\dagger_{-\vec{p}} \Big) e^{i\vec{p}\cdot\vec{x}} \,.$$

a) Show that the commutation relations of the ladder operators,

$$[a_{\vec{p}}, a^\dagger_{\vec{q}}] = (2\pi)^3 \delta^{(3)}(\vec{p} - \vec{q}) \,,$$

lead to the commutation relations $[\varphi(\vec{x}), \pi(\vec{y})] = i\delta^{(3)}(\vec{x} - \vec{y})$ of the field operators.

b) Show that the Hamiltonian

$$H = \int d^3x \left[\frac{1}{2}\pi^2 + \frac{1}{2}(\vec{\nabla}\varphi)^2 + \frac{1}{2}m^2\varphi^2 \right]$$

becomes

$$H = \int \frac{d^3p}{(2\pi)^3} E_{\vec{p}} \Big(a^\dagger_{\vec{p}} a_{\vec{p}} + \frac{1}{2} [a_{\vec{p}}\,, a^\dagger_{\vec{p}}] \Big).$$

c) Show that the total momentum operator

$$\vec{P} = -\int d^3x \, \pi(\vec{x}) \vec{\nabla}\varphi(\vec{x})$$

becomes

$$\vec{P} = \int \frac{d^3p}{(2\pi)^3} \, \vec{p} \, a^\dagger_{\vec{p}} a_{\vec{p}} \,.$$

Problem 2 Lorentz invariance of the normalization

In the lecture we introduced the normalization $|\vec{p}\rangle = \sqrt{2E_{\vec{p}}}\, a^\dagger_{\vec{p}}|0\rangle$, which led to

$$\langle \vec{p} | \vec{q} \rangle = 2E_{\vec{p}} (2\pi)^3 \delta^{(3)}(\vec{p} - \vec{q}) \,.$$

a) Show that the quantity $E_{\vec{p}}\,\delta^{(3)}(\vec{p} - \vec{q})$ is a Lorentz scalar for the special case of a boost in the 3-direction, i.e., $p_3' = \gamma(p_3 + \beta E)$ and $E' = \gamma(E + \beta p_3)$.

b) Show that this quantity is also a scalar under the most general Lorentz transformation.

Problem 3 Time dependence of the momentum density

In the lecture we have shown that $\frac{\partial}{\partial t}\varphi(\vec{x},t) = \pi(\vec{x},t)$. Using the Heisenberg equation of motion and the Hamiltonian given in 1b), show that

$$\frac{\partial}{\partial t}\pi(\vec{x},t) = (\vec{\nabla}^2 - m^2)\varphi(\vec{x},t)\,.$$

Problem 4 3-momentum and 4-momentum operators

Show that

$$e^{-i\vec{P}\cdot\vec{x}} a_{\vec{p}}\, e^{i\vec{P}\cdot\vec{x}} = a_{\vec{p}}\, e^{i\vec{p}\cdot\vec{x}} \qquad \text{and} \qquad e^{-i\vec{P}\cdot\vec{x}} a^\dagger_{\vec{p}}\, e^{i\vec{P}\cdot\vec{x}} = a^\dagger_{\vec{p}}\, e^{-i\vec{p}\cdot\vec{x}}\,,$$

where $\vec{P}$ is the 3-momentum operator. Show that this leads to

$$\begin{aligned}\varphi(x) &= e^{i(Ht-\vec{P}\cdot\vec{x})}\varphi(0)e^{-i(Ht-\vec{P}\cdot\vec{x})}\\ &= e^{iP\cdot x}\varphi(0)e^{-iP\cdot x}\,,\end{aligned}$$

where $P^\mu = (H,\vec{P})$ is the 4-momentum operator.