Universität Regensburg, Institut für Theoretische Physik SS 2024
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Quantum Electrodynamics
Problem Set 1 (for the exercises on April 23 and 24)

Problem 1 Natural units

a) Recall that we are using Heaviside-Lorentz units throughout. Show that in natural units ($\hbar = c = 1$) the Compton wavelength of an electron is m_e^{-1}, the Bohr radius of the hydrogen atom is $(\alpha m)^{-1}$, and the velocity of an electron in its lowest Bohr orbit is simply α. Here, $m = m_e m_p/(m_e + m_p)$ is the reduced mass.

b) Cross sections (which we will get to know later in the course) are often expressed in millibarns, where 1 mb $= 10^{-3}$ b $= 10^{-27}$ cm^2. Show that 1 GeV$^{-2} = 0.389$ mb.

Problem 2 Classical electrodynamics

Classical electrodynamics follows from the action

$$S = \int d^4x\, \mathcal{L} \quad \text{with} \quad \mathcal{L} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} - J^\mu A_\mu \,,$$

where $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$ and $J^\mu = (\rho, \vec{J})$.

a) Derive Maxwell's equations as the Euler-Lagrange equations of this action, treating the components $A_\nu(x)$ as the dynamical variables. Write the equations in standard form by identifying $E^i = -F^{0i}$ and $\varepsilon^{ijk} B^k = -F^{ij}$.

b) Set $J^\mu = 0$ and construct the energy-momentum tensor $T^{\mu\nu}$ for this theory, defined as

$$T^{\mu\nu} = \frac{\partial \mathcal{L}}{\partial(\partial_\mu A_\lambda)} \partial^\nu A_\lambda - \mathcal{L} g^{\mu\nu} \,.$$

Note that the usual procedure does not result in a symmetric tensor. To remedy that, we can add to $T^{\mu\nu}$ a term of the form $\partial_\lambda K^{\lambda\mu\nu}$, where $K^{\lambda\mu\nu}$ is antisymmetric in its first two indices. Such an object is automatically divergenceless, so

$$\widehat{T}^{\mu\nu} = T^{\mu\nu} + \partial_\lambda K^{\lambda\mu\nu}$$

is an equally good energy-momentum tensor with the same globally conserved energy and momentum. Show that this construction, with

$$K^{\lambda\mu\nu} = F^{\mu\lambda} A^\nu \,,$$

leads to an energy-momentum tensor $\widehat{T}$ that is symmetric and yields the standard formulas for the electromagnetic energy and momentum densities,

$$\mathcal{E} = \frac{1}{2}(E^2 + B^2) \quad \text{and} \quad \vec{S} = \vec{E} \times \vec{B} \,,$$

where $\mathcal{E} = \widehat{T}^{00}$ and $S_k = \widehat{T}^{0k}$.

Problem 3 Lack of causality in quantum mechanics

In quantum mechanics, the amplitude for a free particle to propagate from $\vec{x}_0$ to $\vec{x}$ is given by

$$U(t) = \langle \vec{x}| e^{-iHt} |\vec{x}_0\rangle \, .$$

a) In nonrelativistic quantum mechanics, we have $E = \vec{p}^{\,2}/2m$. Show, by inserting a complete set of momentum eigenstates and performing a Gaussian integral, that

$$U(t) = \left(\frac{m}{2\pi i t}\right)^{3/2} e^{imr^2/2t} \, ,$$

where $r = |\vec{x} - \vec{x}_0|$. Note that $U(t)$ is nonzero for all r and t. This means that the particle can propagate between any two points in an arbitrarily short time.

b) In a relativistic theory, the conclusion from part a) would signal a violation of causality. Perhaps using the relativistic expression $E = \sqrt{|\vec{p}|^2 + m^2}$ solves this problem? Show that in this case we obtain

$$U(t) = \frac{1}{2\pi^2 r} \int_0^\infty dp \, p \, \sin(pr) \, e^{-it\sqrt{p^2+m^2}} \, .$$

c) (Bonus) The integral in part b) could be evaluated explicitly in terms of Bessel functions (Gradshteyn & Ryzhik, 5th ed., #3.914). We will not follow this route but instead compute the asymptotic form of $U(t)$ for $r^2 \gg t^2$, which is well outside the light cone. Using the method of stationary phase, show that in this case

$$U(t) \sim e^{-m\sqrt{r^2-t^2}}$$

(up to a rational function of r and t). Thus, $U(t)$ is small but nonzero outside the light cone, and therefore causality is still violated.
Note: If you don't know the method of stationary phase (a.k.a. saddle-point approximation or method of steepest descent), look it up in a book or google it. We will also discuss it in the exercises.