## Quantum Information Theory

Prof. John Schliemann
Dr. Paul Wenk

Tue. H33 13pm c.t. \& Thu. H34, 3pm c.t. Mon. 12pm c.t., H33

## Sheet 9

## 1 Berry's Phase

Let $H(\mathbf{R}(t))$ be the Hamiltonian of a system with an external time-dependent parameter $\mathbf{R}(t) \in \mathbb{R}^{3}$. $\mathbf{R}(t)$ is changed adiabatically. Given further the eigenstates

$$
\begin{equation*}
H(\mathbf{R}(t))|n, \mathbf{R}(t)\rangle=E_{n}(\mathbf{R}(t))|n, \mathbf{R}(t)\rangle \tag{1}
\end{equation*}
$$

we have seen in the lecture that the corresponding Berry's phase is given by

$$
\begin{equation*}
\gamma_{n}(C)=i \int_{C} \mathrm{~d} \mathbf{R} \cdot\langle n, \mathbf{R}| \nabla_{\mathbf{R}}|n, \mathbf{R}\rangle \tag{2}
\end{equation*}
$$

for a closed loop $C$ in the parameter space where $\mathbf{R}$ is living. In the following we are going to show that this geometric phase can be interpreted as a surface integral of a vector $\mathbf{V}_{n}(\mathbf{R})$ penetrating through the surface $S(C)$ surrounded by $C$.
(a) Show that

$$
\begin{equation*}
\gamma_{n}(C)=-\iint_{S(C)} \mathrm{d} \mathbf{S} \cdot \mathbf{V}_{n}(\mathbf{R}) \tag{3}
\end{equation*}
$$

with

$$
\begin{equation*}
\mathbf{V}_{n}(\mathbf{R})=\operatorname{Im} \sum_{n \neq m} \frac{\langle n, \mathbf{R}| \nabla_{\mathbf{R}} H(\mathbf{R})|m, \mathbf{R}\rangle \times\langle m, \mathbf{R}| \nabla_{\mathbf{R}} H(\mathbf{R})|n, \mathbf{R}\rangle}{\left(E_{m}(\mathbf{R})-E_{n}(\mathbf{R})\right)^{2}} \tag{4}
\end{equation*}
$$

(b) Assume a spin of magnitude $S$ in a time-dependent magnetic field $\mathbf{B}(t)$. Further assume that the spin follows the magnetic field adiabatically while $\mathbf{B}(t)$ slowly changes its direction with $t$. The corresponding Hamiltonian is given by $H(\mathbf{B})=-(g \mu / \hbar) \mathbf{S} \cdot \mathbf{B}$. Show that the Berry's phase is given by

$$
\begin{equation*}
\gamma_{m}(C)=-m \iint_{S(C)} \mathrm{d} \mathbf{S}_{\mathbf{B}} \cdot \frac{\mathbf{B}}{B^{3}}=-m \Omega(C) \tag{5}
\end{equation*}
$$

where $\mathbf{B}(t)$ describes a closed path $C$ and $m$ the magnetic quantum number. Explain what $\Omega(C)$ is. Hint: Take the z-axis of the spin space to be in the direction of the instantaneous magnetic field.
(c) How does $\mathbf{V}_{\lambda}(\mathbf{R})$ (with $\lambda$ the according eigenvalue quantum number) look like for a generic two-level system given by

$$
H(\mathbf{R})=H(X, Y, Z)=\frac{1}{2}\left(\begin{array}{cc}
Z & X-i Y  \tag{6}\\
X+i Y & -Z
\end{array}\right) ?
$$

Thus, what is the geometric factor for the level crossing in this system?

## 2 Gauge Invariance

(a) Let a particle be in a potential $V(\mathbf{x})$. Further, let $\psi(\mathbf{x}, t)$ be the solution of the Schrödinger equation describing the motion of this particle. We apply the gauge transformation

$$
\begin{equation*}
\psi(\mathbf{x}, t) \rightarrow \psi^{\prime}(\mathbf{x}, t)=e^{i \alpha} \psi(\mathbf{x}, t) \quad \text { with the constant } \quad \alpha \in \mathbb{R} \tag{7}
\end{equation*}
$$

Obviously, this global gauge transformation does not change the Schrödinger equation, $\psi^{\prime}$ is still a solution. The situation changes under a local gauge transformation,

$$
\begin{equation*}
\psi(\mathbf{x}, t) \rightarrow \psi^{\prime}(\mathbf{x}, t)=e^{i \Lambda(\mathbf{x}, t)} \psi(\mathbf{x}, t) \tag{8}
\end{equation*}
$$

Assuming that $\psi^{\prime}(\mathbf{x}, t)$ is a solution, show that the corresponding transformed equation of motion for $\psi(\mathbf{x}, t)$ is not the Schrödinger equation anymore. Show that to make the Schrödinger equation locally gauge invariant in case of a general local gauge as shown in Eq. (8), we need to have a vector field $\mathbf{A}(\mathbf{x}, t)$ and a scalar field $\phi(\mathbf{x}, t)$. Their presence allows the Schrödinger equation to be invariant under the local gauge transformation which is now given by $\psi(\mathbf{x}, t) \rightarrow \psi^{\prime}(\mathbf{x}, t)=e^{i \Lambda(\mathbf{x}, t)} \psi(\mathbf{x}, t), \phi(\mathbf{x}, t) \rightarrow \phi^{\prime}(\mathbf{x}, t)$ and $\mathbf{A}(\mathbf{x}, t) \rightarrow \mathbf{A}^{\prime}(\mathbf{x}, t)$ with the appropriate fields $\mathbf{A}^{\prime}$ and $\phi^{\prime}$.
(b) Gauge transformations can also be non-abelian. In this case, $\Lambda(\mathbf{x}, t)$ is not a number anymore. As an example let us analyze a simple model given by the following Hamiltonian $H$ which describes a special spin-orbit coupling (SOC) in a two dimensional electron gas,

$$
\begin{equation*}
H=\frac{\hbar^{2}}{2 m^{*}}\left(k^{2}+2(\mathbf{k} \cdot \mathbf{Q}) \Sigma\right) \tag{9}
\end{equation*}
$$

with

$$
\begin{equation*}
\Sigma=\frac{\sigma_{y}-\sigma_{x}}{\sqrt{2}}, \quad \mathbf{Q}=\frac{\sqrt{2} m^{*} \alpha}{\hbar}\binom{1}{1} \tag{10}
\end{equation*}
$$

where $m^{*}$ is the effective electron mass, $\sigma_{i}$ Pauli matrices, $\mathbf{k}$ the wave vector in the $x-y$-plane and $\alpha$ the strength of the SOC.
(i) Show that $H$ can be rewritten in the form

$$
\begin{equation*}
H=\frac{1}{2 m^{*}}(\mathbf{p}+\underbrace{\hbar \mathbf{Q} \Sigma}_{=:-e \mathbf{A}_{\mathrm{SO}}})^{2}-\frac{\hbar^{2} \mathbf{Q}^{2}}{2 m^{*}} \tag{11}
\end{equation*}
$$

where $\mathbf{p}$ can be interpreted as a gauge-dependent canonical momentum.
(ii) Show that $U=\exp (-i \mathbf{Q} \cdot \mathbf{r} \Sigma)$ can be used as a gauge transformation which "gauges away" the vector potential $\mathbf{A}_{\mathrm{SO}}$.

