Quantum Information Theory

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Tue. H33 13pm c.t. & **Thu.** H34, 3pm c.t.
Mon. 12pm c.t., H33

Sheet 6

1 Implementation of a SWAP Gate .... [4P]

Consider the coupling between two electron spins $\mathbf{S}_i$ described by the Heisenberg Hamiltonian,

$$H(t) = J(t)\mathbf{S}_1 \cdot \mathbf{S}_2 \,. \tag{1}$$

Assume the coupling to be switched on at $t = 0$, i.e., $J(t) = J_0\theta(t)$ with the Heaviside function $\theta(.)$. At which moment is the time evolution operator $U(t)$ of H, given an appropriate coupling strength J_0, acting as a swap operator?

2 Magic Gate .... [8P]

This exercise is related to finding an optimal implementation of two-qubit gates. An important matrix M in this context is the *magic gate*, a matrix which transforms the binary basis into the magic basis $|j\rangle_M = M|j\rangle_B$ with $|j\rangle_M \in \{\frac{1}{\sqrt{2}}(|00\rangle + |11\rangle), \frac{i}{\sqrt{2}}(|00\rangle - |11\rangle), \frac{i}{\sqrt{2}}(|01\rangle + |10\rangle), \frac{1}{\sqrt{2}}(|01\rangle - |10\rangle)\}$. The magic basis differs only by phase from an ordinary Bell basis.

(a) Show that $M = U_{\mathrm{CNOT2}}(\mathbb{1}_2 \otimes H)(S \otimes S)$ where H is the Hadamard gate, S the phase gate, and U_{CNOT2} the CNOT gate where qubit 2 is the control qubit and qubit 1 is the target qubit

(b) Show that for every $U \in \mathrm{SO}(4)$ the matrix $MUM^\dagger \in \mathrm{SU}(2) \otimes \mathrm{SU}(2)$. Moreover, this mapping is an isomorphism.
Hint: Recall from Sheet 1 that every unitary matrix can be written using a phase, R_z and R_y. Now, $SU(n) \subset U(n)$ are all unitary matrices with determinant 1. The fact that SO(4) and $SU(2) \otimes SU(2)$ have the same dimension does not need to be proven. Use (a) for the calculation of M^{-1}.

3 QFT .... [6P]

Given three qubits and the orthogonal basis $\{|0\rangle \equiv |000\rangle, |1\rangle \equiv |001\rangle \ldots, |7\rangle \equiv |111\rangle\}$, calculate the Fourier transformed state of

$$|\psi\rangle = \frac{1}{2}\sum_{n=0}^{7} \cos\left(2\pi\frac{n}{8}\right)|n\rangle \tag{2}$$

by applying the quantum Fourier transform.

4 Congruences .... [6P]

Find all solutions x_Λ for $8 < \Lambda < 12$ with

$$x_\Lambda \equiv 4 \pmod{5} \,, \tag{3}$$
$$x_\Lambda \equiv 4 \pmod{7} \,, \tag{4}$$
$$x_\Lambda \equiv 6 \pmod{\Lambda} \tag{5}$$

by applying the Chinese remainder theorem.

5 Composite Numbers . . . . [7P]

Prove that each increasing arithmetic progression[1] of $x \in \mathbb{N}$ contains an arbitrarily long sequence of consecutive terms which are composite numbers[2]. What is the consequence for prime numbers?
Hint: Examine the set of equations $ax \equiv -b - aj \pmod{q_j^2}$, *where* $m > 0$, $j = 1, 2, \ldots, m$, *with primes* q_i *such that* $a < q_1 < q_2 < \ldots < q_m$.

[1] sequence of numbers such that the difference between the consecutive terms is constant
[2] a positive integer that can be formed by multiplying two smaller positive integers