

Quantum Information Theory

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Tue. H33 13pm c.t. & **Thu.** H34, 3pm c.t.
Mon. 12pm c.t., H33

Sheet 4

1 Big-O: Hardness of problems [6P]

- Show that a classical circuit constructed of n NAND gates is able to implement at most $\mathcal{O}(n^{2n})$ boolean functions.
- Now, consider an arbitrary unitary transformation which acts on n qubits. Show that it has $\mathcal{O}(2^{2n})$ degrees of freedom.

2 Grover iteration [6P]

- Which part of the Grover iteration acting on two qubits is represented in Fig. 1?
- Let us continue with the two-bit search. Assume the oracle with $f(x) = 0$ for all x except for $x = x_0$ with $f(x_0) = 1$. With two qubits we have a search space of size $N = 2^2$.
 - Write down the possible oracle circuits.
 - How many iterations n are needed in this case to get to the solution state? Consult the upper bound formula and apply the Grover iteration G^n explicitly on the initial state.
 - How many queries are required on average in the classical case?

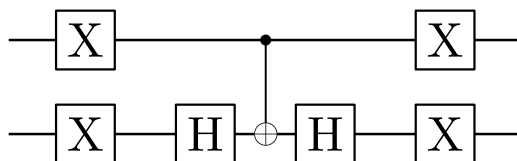


Figure 1: Part of the quantum circuit for the Grover iteration.

3 Arbitrary Single Qubit Unitary Operator [10P]

- Given are two arbitrary unit vectors $\hat{\mathbf{n}}$ and $\hat{\mathbf{m}}$ with $\hat{\mathbf{m}} \times \hat{\mathbf{n}} \neq \mathbf{0}$. Is it possible to write an arbitrary single qubit unitary operator U as

$$U = e^{i\alpha} R_{\hat{\mathbf{n}}}(\beta) R_{\hat{\mathbf{m}}}(\gamma) R_{\hat{\mathbf{n}}}(\delta) \quad \text{with} \quad R_{\hat{\mathbf{n}}}(\phi) := \exp(-i\phi \hat{\mathbf{n}} \cdot \boldsymbol{\sigma}/2), \quad \alpha, \beta, \gamma, \delta, \phi \in \mathbb{R} \quad (1)$$

Hint: Read paper by Mitsuru Hamada.

- Show that $X R_{\hat{\mathbf{y}}}(\theta) X = R_{\hat{\mathbf{y}}}(-\theta)$

- (c) Given a single qubit unitary operator U , show that U can be represented using three unitary operators A, B, C on a single qubit with

$$ABC = \mathbb{1} \quad \text{and} \quad U = e^{i\alpha} AXBXC \quad \text{with} \quad \alpha \in \mathbb{R} . \quad (2)$$

Hint: Choose A as a composition of rotations R_z and R_y , B as a composition of R_y and R_z and C as R_z with appropriate angles.

- (d) Write the Hadamard gate H and the phase gate S in the form of Eq. (1).
 (e) Reconstruct the quantum circuit shown in Fig. 2 with

- (i) $U = R_x(\theta)$,
- (ii) $U = R_y(\theta)$,
- (iii) $U = R_k$ which is needed in the quantum Fourier transform,

by using only CNOT and single qubit gates.

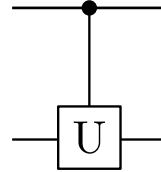


Figure 2: Quantum circuit of a controlled U .
