

Quantum Information Theory

Prof. John Schliemann
Dr. Paul Wenk

Tue. H33 13pm c.t. & **Thu.** H34, 3pm c.t.
Mon. 12pm c.t., H33

Sheet 3

1 Tsirelson's Inequality [6P]

Let $\mathbf{Q} = \mathbf{e}_1 \cdot \boldsymbol{\sigma}$, $\mathbf{R} = \mathbf{e}_2 \cdot \boldsymbol{\sigma}$, $\mathbf{S} = \mathbf{e}_3 \cdot \boldsymbol{\sigma}$, and $\mathbf{T} = \mathbf{e}_4 \cdot \boldsymbol{\sigma}$ where $\mathbf{e}_i \in \mathbb{R}^3$ are arbitrary unit vectors. Following the lecture notes, show explicitly the inequality

$$\langle \mathbf{Q} \otimes \mathbf{S} \rangle + \langle \mathbf{R} \otimes \mathbf{S} \rangle + \langle \mathbf{R} \otimes \mathbf{T} \rangle - \langle \mathbf{Q} \otimes \mathbf{T} \rangle \leq 2\sqrt{2}. \quad (1)$$

2 Von Neumann Entropy [6P]

The von Neumann entropy of a system A is defined by $S(A)_{\rho_A} = -\text{tr}(\rho_A \log(\rho_A)) = -\sum_i \lambda_i \log(\lambda_i)$ with eigenvalues λ_i of ρ_A . Respectively, the von Neumann entropy of a composite system AB is defined by $S(AB)_{\rho_{AB}} = -\text{tr}(\rho_{AB} \log(\rho_{AB})) = -\sum_i \lambda_i \log(\lambda_i)$ with the corresponding eigenvalues of ρ_{AB} .

(I) Prove the following statements:

- (a) For every ρ_A : $S(A)_{\rho_A} \geq 0$.
- (b) If ρ_{AB} is pure, then $S(A)_{\rho_A} = S(B)_{\rho_B}$.
- (c) If $\rho_{AB} = \rho_A \otimes \rho_B$, then $S(AB)_{\rho_{AB}} = S(A)_{\rho_A} + S(B)_{\rho_B}$.

(II) Given the Bell state $|B_{00}\rangle = (|00\rangle + |11\rangle)/\sqrt{2}$, compute $S(A)$ and $S(AB)$ where A labels the first qubit and B the second one.

3 Superdense Coding [6P]

Recall the superdense coding procedure discussed in the lecture. Assume that an eavesdropper intercepts Alice's qubit on the way to Bob. Is it possible for him to infer which two bits Alice sent to Bob?

Hint: Analyze the expectation value of $\Lambda \otimes \mathbb{1}$ in the Bell states for a positive operator Λ .

4 Generalized Deutsch Algorithm [6P]

Recall Ex. (3) from sheet 2. Here, we generalize the bit function to $f(x) : \{0, 1\}^n \rightarrow \{0, 1\}$.

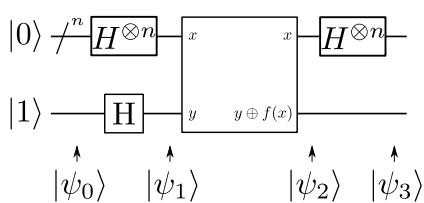


Figure 1: Quantum circuit of Ex. 4

(a) Write down the quantum states $|\psi_i\rangle$ of the circuits according to Fig. 1.

(b) Given the n qubits of the top register are measured: Which result do we obtain...

- ...if $f(x)$ is balanced?
- ...if $f(x)$ is constant?

5 Carrier Bit [6P]

Construct a quantum circuit which takes x and y as input bits and returns the sum $x \oplus y$ with c as the carrier bit, i.e., $(x, y, c, x \oplus y)$.

- Plot the circuit.
- Write down the corresponding tensor product of the operation.
- Bonus:* Generate the corresponding matrix in *Mathematica* (or some other prog.) of the circuit in the computational basis and let it do the calculation for you.

6 CNOT Flip [6P]

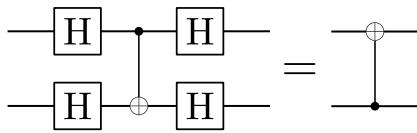


Figure 2: Quantum circuit of Ex. 6

- Show that the equivalence shown in Fig. 2 holds.

Hint: Write down CNOT as a sum of tensor products of $\mathbb{1}$ and σ_i , combine it with H and apply $(A \otimes B).(C \otimes D) = A.C \otimes B.D$.

- Apply the circuit to the states $|\pm\pm\rangle$, $|\pm\mp\rangle$ where $|\pm\rangle = (|0\rangle \pm |1\rangle)/\sqrt{2}$. How did the functionality of the CNOT change in the new basis?
-