Quantum Information Theory

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Tue. H33 13pm c.t. & **Thu.** H34, 3pm c.t.
Mon. 12pm c.t., H33

Sheet 2

1 Hadamard operator II .... [6P]

Show that the Hadamard operator which transforms n qubits, $H^{\otimes n} := \bigotimes_{i=1}^{n} H_i$, can be written as

$$H^{\otimes n} = \frac{1}{\sqrt{2^n}} \sum_{\mathbf{a},\mathbf{b}} (-1)^{\mathbf{a}\cdot\mathbf{b}} |\mathbf{a}\rangle\langle\mathbf{b}| \quad \text{with} \quad |\mathbf{a}\rangle, |\mathbf{b}\rangle \in \left\{ \bigotimes_{i=1}^{n} |\sigma_i\rangle \,\middle|\, \sigma_i \in \{0,1\} \right\}. \tag{1}$$

2 Teleportation .... [6P]

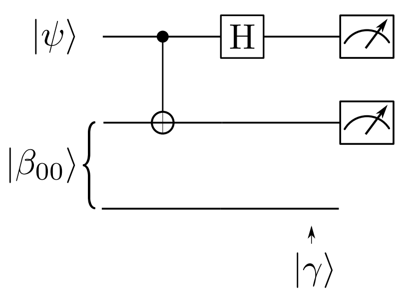

Figure 1: Quantum circuit of Ex. 2

A part of the quantum circuit for teleportation of a normalized state $|\psi\rangle = \alpha|0\rangle + \beta|0\rangle$, $\alpha, \beta \in \mathbb{C}$ by Alice to Bob is shown in Fig. 1. The two top lines represent Alice's qubits. Bob's qubit is one qubit of the Bell state $|\beta_{00}\rangle = (|00\rangle + |11\rangle)/\sqrt{2}$. The other qubit is kept by Alice.

(a) Write down the three qubit state $|\gamma\rangle$ which is reached after the application of the Hadamard gate (see Fig. 1). Measuring in Alice's computational basis, what are the probabilities of the states of the system after the measurement as depicted in Fig. 1?

(b) Write down the density matrix of the system after Alice performed her measurement and trace out Alice's system. Has Bob now some information about $|\psi\rangle$ without having contacted Alice?

3 Deutsch Algorithm .... [6P]

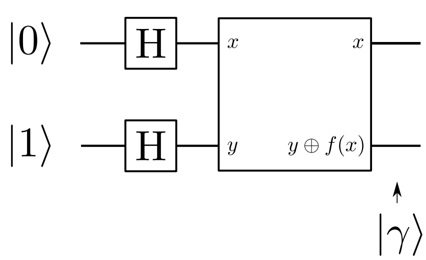

Figure 2: Quantum circuit of Ex. 3

Show that the two qubit state $|\gamma\rangle$ in Fig. 2 is given by

$$|\gamma\rangle = \begin{cases} \pm\left[\frac{|0\rangle+|1\rangle}{\sqrt{2}}\right]\left[\frac{|0\rangle-|1\rangle}{\sqrt{2}}\right] & : \quad f(0) = f(1) \\ \pm\left[\frac{|0\rangle-|1\rangle}{\sqrt{2}}\right]\left[\frac{|0\rangle-|1\rangle}{\sqrt{2}}\right] & : \quad f(0) \neq f(1) \end{cases}, \tag{2}$$

where $f(x) : \{0,1\} \to \{0,1\}$.

4 Double Quantum Well ....[12P]

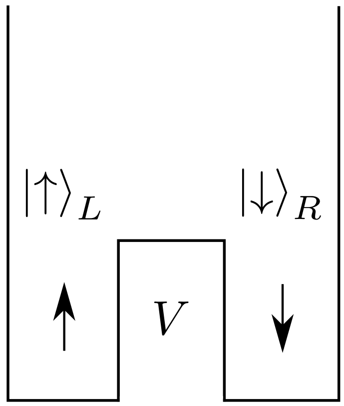

Figure 3: Double quantum well, Ex. 4

Consider a double quantum well with infinite potentials on the left and right borders, see Fig. 3, containing two electrons. Assume the initial state to be $|\uparrow\rangle_L \otimes |\downarrow\rangle_R =: |\uparrow\downarrow\rangle$ where $\{|\uparrow\rangle, |\downarrow\rangle\}$ are the eigenstates of σ_z. Assume the dynamics of the system to be governed by the Hamiltonian

$$H = \sum_{j\sigma} \epsilon c^\dagger_{j\sigma} c_{j\sigma} + \sum_\sigma b(c^\dagger_{1\sigma} c_{2\sigma} + c^\dagger_{2\sigma} c_{1\sigma}), \quad \text{with} \quad \sigma \in \{\uparrow, \downarrow\} \,. \tag{3}$$

The operator $c^\dagger_{j\sigma}$ creates an electron on site $j \in \{1 \equiv L, 2 \equiv R\}$.

(a) Diagonalize H.
Hint: H is invariant under parity P $(1 \leftrightarrow 2)$ and spin flip S_F $(\uparrow\leftrightarrow\downarrow)$. Re-organize the two particle states according to these symmetries to build a more suitable basis.

(b) Given the initial state $|\uparrow\downarrow\rangle$ at $t = 0$, write down the time dependent density matrix of the total system $\hat{\rho}(t)$.

(c) Calculate the time dependent reduced density matrix $\hat{\rho}_{\text{red}}(t)$ which is defined by tracing out one of the sites of the quantum well. With this, show that $\hat{\rho}_{\text{red}}(t)$ is a statistical mixture except for $t = n\pi/\omega$ where $n \in \mathbb{Z}$ and $\omega = 2b/\hbar$.