University of Regensburg WS 2019/20

Quantum Information Theory

Prof. John Schliemann **Tue.** H33 13pm c.t. & **Thu.** H34, 3pm c.t.
Dr. Paul Wenk **Mon.** 12pm c.t., H33

Sheet 11

1 3-Qubit Code . . . . [10P]

Consider three qubits which are exposed to errors. The errors are rotations of the form $U = \exp(i\epsilon\sigma_x)$ with $\epsilon \ll 1$. Each qubit can be affected, thus

$$|\psi\rangle_E = E\,|\psi\rangle_L \quad \text{with} \quad E = U^{\otimes 3}\,. \tag{1}$$

The corresponding quantum circuit for encoding an arbitrary state $|\psi\rangle = \alpha\,|0\rangle + \beta\,|1\rangle$ with $\alpha, \beta \in \mathbb{C}$, $|\alpha|^2 + |\beta|^2 = 1$, and correcting the error is shown in Fig. 1.

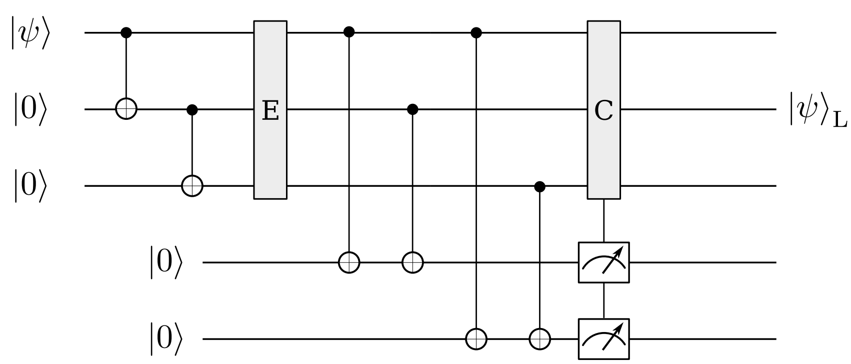

Figure 1: Quantum circuit for encoding and correcting (C-operation) a single bit-flip error (E-operation).

(a) Write down the ancilla measurements for a single σ_x error and the ambiguity of the syndrome results when multiple errors occur.

(b) Now, assume that the errors acting on the qubits are described by U. Corresponding to the measured ancilla block an appropriate correction operator is applied (see Fig. 1 and Tab. 1). Compare the fidelity $F_0 = |\langle\psi|U|\psi\rangle|^2$ without error correction code with the worst case fidelity (= state $|\psi\rangle_L$ is orthogonal to the state $\sigma_x^{\otimes 3}|\psi\rangle_L$) after applying the error correction.

ancilla	correction
00	$c_0\,|\psi\rangle_L + c_3\sigma_x^{\otimes 3}\,|\psi\rangle_L$
01	$c_1\,|\psi\rangle_L + c_2\sigma_x^{\otimes 3}\,|\psi\rangle_L$
10	$c_1\,|\psi\rangle_L + c_2\sigma_x^{\otimes 3}\,|\psi\rangle_L$
11	$c_1\,|\psi\rangle_L + c_2\sigma_x^{\otimes 3}\,|\psi\rangle_L$

Table 1: Quantum states after error correction. $c_0 = \cos^3(\epsilon)$, $c_1 = i\cos^2(\epsilon)\sin(\epsilon)$, $c_2 = -\cos(\epsilon)\sin^2(\epsilon)$, $c_3 = -i\sin^3(\epsilon)$.

2 Quantum operations . . . . [10P]

In the lecture the *operator-sum representation* has been used to represent quantum operations as $\sum_n E_n \rho E_n^\dagger$ using the *operation elements* $\{E_i\}$.

(a) To get to this representation one assumes that the environment starts in a pure state. This can be done w.l.o.g. because we can *purify* a mixed state by introducing an additional system. To see this, consider two systems A and B with $\rho_A = \sum_n \lambda_n |a_n\rangle\langle a_n|$.

(i) Show that $|\psi\rangle = \sum_n \sqrt{\lambda_n}\, |a_n\rangle_A \otimes |b_n\rangle_B$ is a *purification* of ρ_A for *any* orthonormal basis $\{|b_n\rangle\}_n$ with $\rho_A = \mathrm{tr}_B(|\psi\rangle\langle\psi|)$.

(ii) The purification of the system is not unique. Consider a second purifying system B'. Show that B and B' are related via a unitary transformation.
Hint: Write down the purified system using the Schmidt decomposition.

(b) Consider the quantum circuit shown in Fig. 2. What are the corresponding operation elements E_n and what is the physical process described by this system?

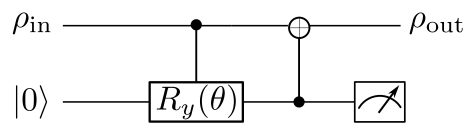

Figure 2: Quantum circuit

3 Generator Matrix and Hamming Code . . . . [8P]

Consider the generator matrix over a finite field of order 3[1]

$$G = \begin{pmatrix} 0 & 1 & 2 & 1 \\ 1 & 0 & 1 & 0 \\ 1 & 2 & 2 & 1 \end{pmatrix} . \tag{2}$$

Calculate the corresponding *parity check matrix* H.
Hint: Transform G into the standard form, i.e., $G = [\mathbb{1}\ P]$.

[1] also called Galois field, here GF(3); the coset $\mod 2$ is GF(2)