## Quantum Information Theory

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## Sheet 10

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# 1  Fidelity . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . [10P]

The *fidelity* is defined as $F(\rho_1, \rho_2) := \mathrm{tr}\left[\sqrt{\rho_1^{1/2}\,\rho_2\,\rho_1^{1/2}}\,\right]^2$, where $\rho_i$ are density matrices.

(a)  (i)  Given a qubit in an unknown state $|\psi\rangle$ calculate the fidelity of a random guess $|\phi\rangle$.

    (ii)  Now, show that the fidelity *improves* if a measurement has been applied to the qubit. To see this, calculate the *average fidelity* of the guess after the measurement.

(b)  Prove that the fidelity is invariant under unitary transformations, i.e.,

$$F(U\rho_1 U^\dagger, U\rho_1 U^\dagger) = F(\rho_1, \rho_2)\,. \tag{1}$$

# 2  Simple Error Correction . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . [10P]

Consider (as in the lecture) three qubits which are exposed to errors. This errors can be described by a linear combination (with coefficients lying in $\mathbb{C}$) of the following unitary matrices,

$$\{U_{\mathrm{CNOT}} U_{\mathrm{CNOT}} \otimes \mathbb{1}_2,\ \mathbb{1}_2 \otimes U_N \otimes U_N,\ \mathbb{1}_2 \otimes U_P \otimes U_P,\ \mathbb{1}_2 \otimes (U_P U_N) \otimes (U_P U_N)\}\,, \tag{2}$$

with $U_N := |0\rangle\langle 1| + |1\rangle\langle 0|$, $U_P := |0\rangle\langle 0| - |1\rangle\langle 1|$, and $U_{\mathrm{CNOT}} := |0\rangle\langle 0| \otimes \mathbb{1}_2 + |1\rangle\langle 1| \otimes X$, where $X = |0\rangle\langle 1| + |1\rangle\langle 0|$. Consider the three-qubit state

$$|\Lambda\rangle = (|00\rangle + |11\rangle) \otimes \psi \quad \text{with} \quad \psi = \alpha\,|0\rangle + \beta\,|1\rangle,\ |\alpha|^2 + |\beta|^2 = 1,\ \alpha, \beta \in \mathbb{C}\,. \tag{3}$$

How can an arbitrary error acting on $\Lambda$ be corrected to regain the correct $|\psi\rangle$?

# 3  Robust against Errors . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . [6P]

Consider errors of the form $E = \exp(-i\phi\sigma_z/2)$ with $\phi \in \mathbb{R}$. Show that encoding $|0\rangle$ and $|1\rangle$ such that

$$|0\rangle \mapsto |0\rangle_c = \frac{1}{\sqrt{2}}(|01\rangle - |10\rangle)\,, \tag{4}$$

$$|1\rangle \mapsto |1\rangle_c = \frac{1}{\sqrt{2}}(|01\rangle + |10\rangle)\,, \tag{5}$$

makes an arbitrary superposition $\alpha\,|0\rangle_c + \beta\,|1\rangle_c$ robust against errors of the form $E \otimes E$.

# 4 Shor's 9 qubit code.....................................[8P]

(a) In the lecture the observables $\tau_1\tau_2$ and $\tau_2\tau_3$ with $\tau_1 = X_1X_2X_3$, $\tau_2 = X_4X_5X_6$, $\tau_3 = X_7X_8X_9$ have been introduced. Show that the syndrome measurement for detecting phase flip errors in Shor's code corresponds to measuring this observables.

(b) Construct a single qubit quantum operations model $\epsilon(\rho) = \sum_n E_n\rho E_n^\dagger$ for quantum noise which replaces every $\rho$ with a completely randomized state $\mathbb{1}/2$. Even this can be corrected by Shor's code!