Computability and ℓ^2-Betti Numbers

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0 Introduction

In Algebraic Topology, a central invariant are the Betti numbers which were introduced by Enrico Betti in 1871, considered by many to be the advent of Algebraic Topology. They are a homotopy invariant that is 'finer' than the Euler characteristic which can be deduced from the Betti numbers via the formula

$$\chi(X) = \sum_{n \in \mathbb{N}} (-1)^n \cdot b_n(X)$$

(where X is a finite CW complex). Moreover, they satisfy the *Künneth formula*, i.e. for finite CW complexes X, Y and $n \in \mathbb{N}$, we have

$$b_n(X \times Y) = \sum_{p+q=n} b_p(X) \cdot b_q(Y).$$

However, one disadvantage is that there is no 'nice' behaviour with respect to finite coverings. This is in contrast to the Euler characteristic χ: For a d-sheeted covering $Y \to X$ of finite CW-complexes, we have $\chi(Y) = d \cdot \chi(X)$. The corresponding property for Betti numbers is not true (see e.g. self-coverings of S^1).

We attempt to rectify this shortcoming by introducing a stable version of the Betti numbers, called the *ℓ^2-Betti numbers*. Let X be a finite, connected CW-complex, and $x_0 \in X$, such that the fundamental group $G := \pi_1(X, x_0)$ is residually finite. The latter means that there exists a *residual chain*, i.e. a sequence $(G_k)_{k \in \mathbb{N}}$ of finite index, normal subgroups in G, such that $G_k \supset G_{k+1}$ for all $k \in \mathbb{N}$ and

$$\bigcap_{k \in \mathbb{N}} G_k = \{e\}.$$

Note that in many examples, this condition is satisfied. Recall that for all $k \in \mathbb{N}$, we then have a regular covering $p_k : X_k \to X$ such that for $\widetilde{x}_0 \in p^{-1}(x_0)$, we have

$$(p_k)_* \pi_1(X_k, \widetilde{x}_0) = G_k.$$

The number of sheets of p_k is then given by the index $[G : G_k]$. For $n \in \mathbb{N}$, we then define the *n-th ℓ^2-Betti number of X* by

$$b_n^{(2)}(X) := \lim_{k \to \infty} \frac{b_n(X_k)}{[G : G_k]}.$$

We then call the set of all ℓ^2-Betti numbers of CW complexes with a given fundamental group G the set of *ℓ^2-Betti numbers arising from G*.

Note that this is not the usual way to introduce ℓ^2-Betti numbers but by Lück's approximation theorem (Theorem 1.3.9), these two definitions coincide for CW complexes with residually finite fundamental groups and the above notion is well-defined and independent of the chosen residual chain. We will define a more general notion in Section 1.2.

Originally, the ℓ^2-Betti numbers were introduced by Atiyah in 1976 [Ati76] who was studying elliptic equations on non-compact manifolds. Atiyah gave an analytic definition involving the heat kernel on the universal covering of a manifold. In Section 1.2, we will follow yet another approach and use *Combinatorial Laplacians* of the $\mathbb{C}G$-chain complex to define ℓ^2-Betti numbers. For a more leisurely introduction, see an article by Kammeyer and Sauer [KS20].

Now, as it turns out, the ℓ^2-Betti numbers are again a homotopy invariant, still satisfy the adapted version of the *Künneth formula* and we have the *Euler-Poincaré formula*, i.e. for every finite, connected CW complex X, we have

$$\chi(X) = \sum_{n\in\mathbb{N}} (-1)^n \cdot b_n^{(2)}(X).$$

Moreover, we have the *proportionality principle*: Let $p : Y \to X$ be a regular, finite covering with $d \in \mathbb{N}$ sheets, X and Y be finite, connected CW-complexes. Then, for all $n \in \mathbb{N}$, we have

$$b_n^{(2)}(Y) = d \cdot b_n^{(2)}(X).$$

But, there is a price to pay: The (ordinary) Betti numbers are always natural numbers. We can no longer expect this. A priori, ℓ^2-Betti numbers are non-negative *real* numbers. Thus, Atiyah asked the question whether these numbers are always rational or even integral [Ati76]. This question, known as *Atiyah's question* or *Atiyah's conjecture* remained open for more than 30 years until it was solved by Austin [Aus13], Lehner and Wagner [LW13], Grabowski [Gra14] as well as Pichot, Schick and Zuk [PSZ15].

Theorem 1.3.3. *Every non-negative number is an ℓ^2-Betti number of some CW complex with a group action (in the broad sense of Section 1.2).*

This thus gives a negative answer to Atiyah's question. Note however, that there is a positive answer for large classes of groups (see for instance Theorem 1.3.7).

We will now make a step back: Are the ℓ^2-Betti numbers contained in a certain class of real numbers, given some additional assumption? In the above setting of finite CW complexes, this is certainly the case: As CW complexes can be approximated by finite simplicial complexes [Hat02, Theorem 2C.5], which can be described by finite combinatorial data, there are only countable many homotopy types of CW complexes. Hence, also the ℓ^2-Betti numbers arising from these CW complexes are countable.

In order to find a suitable class of real numbers that the ℓ^2-Betti numbers are confined to, we investigate similar results for other 'stable' invariants. The stable commutator length and the simplicial volume were recently proved to be right-computable under suitable assumptions (see Theorem 2.3.1 and Theorem 2.3.9). Here, a number $x \in \mathbb{R}$ is *right-computable* if there is a Turing machine that produces a monotonically decreasing sequence of rational numbers converging to x.

The leading question of this thesis will thus be: Do there exist (sensible) hypotheses such that the ℓ^2-Betti numbers are always right-computable? As it turns out, in many cases, we can even prove *effective computability* (which is a stronger statement).

Organisation of this Thesis

In Chapter 1, we will introduce ℓ^2-Betti numbers after covering the preliminaries for this definition. We will also show an algebraic characterisation of ℓ^2-Betti numbers: For a group G, all ℓ^2-Betti numbers arising from G are given as $\dim_{\mathcal{R}G} \ker(\cdot A)$ for some self-adjoint $A \in M_{n\times n}(\mathbb{Z}G)$ (see Section 1.2.4). In Section 1.3, we will cover Atiyah's conjecture and Lück's approximation theorem.

Chapter 2 is dedicated to the introduction of computability concepts. After a 'naive' introduction into this subject, we will define different computability classes such as EC (effectively computable), LC (left-computable) and RC (right-computable). We will then take a look at some results on right-computability of topological invariants (Section 2.3).

The main part of this thesis is Chapter 3. We will start with a survey on some known computability results on ℓ^2-Betti numbers (Section 3.1). We will then discuss right-, left- and effective computability of ℓ^2-Betti numbers under different assumptions.

Finally, in Chapter 4, we will discuss an implementation of some of the main results in the Lean Theorem Prover. This formally verifies some of these results. The `.lean` files used for this can be found on the DVD that is attached to this thesis, or alternatively in a `git` repository online. More information on how to install these files can be found in Section 4.2.

Main Results

In the main part of this thesis, we prove the following theorems (the formulations are slightly modified to require less knowledge on the specific notations).

Theorem 3.2.1. *Let G be a finitely generated group with solvable word problem. Then, all ℓ^2-Betti numbers arising from G are right-computable.*

From Theorem 3.2.9 we then obtain the following corollary.

Corollary 3.2.11. *Let G be a finitely generated group with solvable word problem. Moreover, let $A \in M_{n\times n}(\mathbb{Z}G)$ be self-adjoint. Then, the following are equivalent:*

1. *The number $\dim_{\mathcal{R}G} \ker(\cdot A)$ is left-computable*
2. *The number $\dim_{\mathcal{R}G} \ker(\cdot A)$ is effectively computable.*
3. *There exists a computable sequence $(\epsilon_k)_{k\in\mathbb{N}}$ of rational numbers such that $\lim_{k\to\infty} \epsilon_k = 0$ and for all $k \in \mathbb{N}_{>0}$, we have*

$$\mu_A\Big(\Big(0, \frac{1}{k}\Big)\Big) \le \epsilon_k,$$

where we denote by μ_A the spectral measure of A (see Definition 1.3.15).

We prove more concrete instances of this result.

Theorem 3.3.1. *Let G be a finitely generated, sofic group (see Definition 3.3.3) with solvable word problem. Then, all ℓ^2-Betti numbers arising from G are effectively computable.*

Theorem 3.3.23. *Let G be a finitely generated group with solvable word problem, and $A \in M_{n\times n}(\mathbb{Z}G)$ be self-adjoint and of determinant class (see Definition 3.3.20). Then, $\dim_{\mathcal{R}G} \ker(\cdot A)$ is effectively computable.*

Theorem 3.4.11. *Let G be a finitely presented, residually finite group (see Definition 1.3.10). Then, all ℓ^2-Betti numbers arising from G are effectively computable.*

For the last result, we give a proof that does *not* rely on Corollary 3.2.11. Instead, we use a sequence arising from Lück's approximation theorem (Theorem 1.3.9) to obtain effective computability.

With the exception of Theorem 3.3.1, there exists a (partial) implementation of these results in the Lean Theorem Prover (see Chapter 4). This implementation provides a proof of concept on how proof assistants can be used to verify theorems even in such 'complex' situations where a lot of theoretical background is needed.

Conventions

Convention 0.0.1. The set of natural numbers is denoted by $\mathbb{N}$ and contains 0. We denote by $\mathbb{Z}$, $\mathbb{Q}$, $\mathbb{R}$, and $\mathbb{C}$ the sets of integers, and rational, real and complex numbers, respectively.

Convention 0.0.2. If S is a (finite or infinite) set, we denote by $F(S)$ the free group generated by S. As a set, this is the set of all finite, reduced words in S and their inverses.

Convention 0.0.3. If G is a group, we denote the trivial element in G by e_G or e.

Convention 0.0.4. Let R be a (unital, not necessarily commutative) ring and $n \in \mathbb{N}$. We denote by $M_{n\times n}(R)$ the ring of $n \times n$-matrices over R.

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Finally, I want to express my gratitude towards my family and friends for their emotional support.

1 Introduction to ℓ^2-Betti Numbers

1.1 Group von Neumann Algebras, Traces and Dimension

We start with an introduction to von Neumann dimensions. Throughout this section, let G be a (discrete, countable) group.

1.1.1 Why we need the von Neumann Algebra

The basic idea leading to ℓ^2-Betti numbers is passing from a connected CW complex X to its universal covering $\widetilde{X}$ and considering homology of this universal covering. As in the ordinary case, we want to define the Betti numbers to be the dimensions of homology. Unfortunately, often $H_n(\widetilde{X}, \mathbb{C})$ is *not* a finite-dimensional vector space over $\mathbb{C}$. However, in reasonable cases, $H_n(\widetilde{X}, \mathbb{C})$ is finitely generated as a $\mathbb{C}G$-module, where $\mathbb{C}G$ is the group ring (see Definition 1.1.2) of the fundamental group $G := \pi_1(X)$ which acts on $\widetilde{X}$ via Deck transformations. We can thus have the idea to develop a dimension theory $\dim_{\mathbb{C}G}$ of finitely generated $\mathbb{C}G$-modules.

The point of this section is to show that this attempt is doomed to fail so that we need a more advanced idea. We follow the example given by Kammeyer [Kam19, p. 1].

For a dimension theory, we would expect at least the following three properties:

1. Non-negativity, i.e. $\dim_{\mathbb{C}G} M \geq 0$ for every (suitable) finitely generated $\mathbb{C}G$-module M.
2. Normality: We have $\dim_{\mathbb{C}G} \mathbb{C}G = 1$.
3. Additivity: If
$$0 \longrightarrow U \longrightarrow V \longrightarrow W \longrightarrow 0$$
is a short exact sequence, we have $\dim_{\mathbb{C}G} V = \dim_{\mathbb{C}G} U + \dim_{\mathbb{C}G} W$.

We show that for $G := F_2$, the free group on two generators, such a dimension theory does not exist. Note that F_2 is the fundamental group of the wedge $X := S^1 \vee S^1$. Its universal covering is a regular 4-tree (see Figure 1.1), which therefore has a natural action by F_2. Because $S^1 \vee S^1$ has one 0-cell and two 1-cells, we obtain $C_0(\widetilde{X}, \mathbb{C}) \cong \mathbb{C}[F_2]$ and $C_1(\widetilde{X}, \mathbb{C}) \cong (\mathbb{C}[F_2])^2$. Note that, canonically identified, the two factors of $C_1(\widetilde{X}, \mathbb{C})$ correspond to the 'horizontal' and 'vertical' edges, respectively.

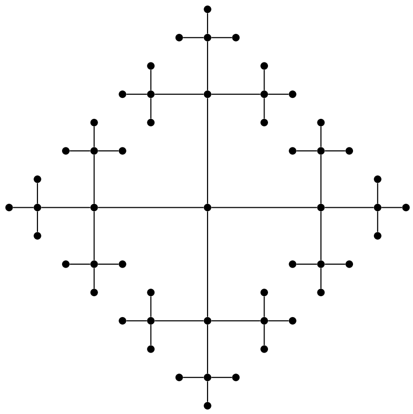

Figure 1.1: the universal covering of $S^1 \vee S^1$, an infinite 4-regular tree

Now, the differential $d_1 : C_1(\widetilde{X}, \mathbb{C}) \to C_0(\widetilde{X}, \mathbb{C})$ sends any edge to the difference of its end points. It turns out that this map is injective as for a non-trivial chain, we can find an 'extremal' point. Therefore, we have a short exact sequence

$$0 \longrightarrow (\mathbb{C}[F_2])^2 \longrightarrow \mathbb{C}[F_2] \longrightarrow \operatorname{coker} d_1 \longrightarrow 0$$

Together with the three conditions on $\dim_{\mathbb{C}G}$, this would imply

$$1 = 2 + \dim_{\mathbb{C}G} \operatorname{coker} d_1,$$

contradicting the non-negativity of the dimension. Thus, such a notion of $\dim_{\mathbb{C}G}$ cannot exist.

We can solve this problem by passing to the completion $\ell^2 G$ of $\mathbb{C}G$. This approach will be developed in the following sections.

1.1.2 Group von Neumann Algebras

In this section, we will introduce the group von Neumann algebra. We will mainly follow Kammeyer's book [Kam19, Chapter 1.2]. We assume knowledge about basic facts on Hilbert spaces and operators, as for instance treated in Kammeyer's book.

Notation 1.1.1 (bounded linear operators)**.** If H is a Hilbert space, we denote by $B(H)$ the space of bounded linear operators $H \to H$.

Definition 1.1.2 (group ring)**.** Let R be a ring. Then, we define the group ring RG as the set of families $(a_g)_{g \in G}$ in R such that for only finitely many $g \in G$, we have $a_g \neq 0$. We write the family $(a_g)_{g\in G}$ as a formal sum $\sum_{g\in G} a_g \cdot g$. We define the addition by

$$\left(\sum_{g\in G} a_g \cdot g\right) + \left(\sum_{g\in G} b_g \cdot g\right) := \sum_{g\in G} (a_g + b_g) \cdot g$$

and a product by

$$\left(\sum_{g\in G} a_g \cdot g\right) \cdot \left(\sum_{g\in G} b_g \cdot g\right) := \sum_{g\in G} \Big(\sum_{h\in G} a_h \cdot b_{h^{-1}\cdot g}\Big) \cdot g$$

Remark 1.1.3. The group ring is, with respect to the defined addition and multiplication, in fact a ring. The unit element is given by $1 \cdot e$, where $e \in G$ denotes the neutral element in G.

Lemma 1.1.4. *The complex group ring $\mathbb{C}G$ carries an inner product given by*

$$\left\langle \sum_{g\in G} a_g \cdot g, \sum_{g\in G} b_g \cdot g \right\rangle := \sum_{g\in G} \overline{a_g} \cdot b_g \in \mathbb{C}$$

where $\bar{\cdot}$ denotes complex conjugation.

Lemma 1.1.5 ($\ell^2 G$ [Kam19, Example 2.22])**.** *The Hilbert space completion of $\mathbb{C}G$ is given by*

$$\ell^2 G := \Big\{ (a_g)_{g\in G} \mid \forall_{g\in G}\ a_g \in \mathbb{C}, \sum_{g\in G} |a_g|^2 < \infty \Big\}$$

where the inner product is given by

$$\langle (a_g)_{g\in G}, (b_g)_{g\in G} \rangle := \sum_{g\in G} \overline{a_g} \cdot b_g \in \mathbb{C}$$

Definition 1.1.6 (left and right regular representations)**.** Note that a group element $h \in G$ acts unitarily on $\ell^2 G$ by $g \mapsto hg$ and also by $g \mapsto gh^{-1}$. We thus obtain left and right actions of $\mathbb{C}G$ on $\ell^2 G$.

From these actions, we obtain the *left regular representation* $\lambda : \mathbb{C}G \hookrightarrow B(\ell^2 G)$ and the *right regular representation* $\rho : \mathbb{C}G \hookrightarrow B(\ell^2 G)$, turning $B(\ell^2 G)$ into a left (resp. right) $\mathbb{C}G$-module.

Remark 1.1.7 (involution). There is a canonical involution on $B(\ell^2 G)$ given by sending an operator to its adjoint. We denote this involution by $T \mapsto T^*$. This defines indeed an involution, as for $S, T \in B(\ell^2 G)$, we have the following identities:

1. $(ST)^* = T^*S^*$.
2. $(T^*)^* = T$.

We thus call $B(\ell^2 G)$ a $*$-*algebra*.

In order to define the group von Neumann algebra, we need to introduce the strong and weak topologies.

Definition 1.1.8 (operator, strong and weak topology)**.** Let H be a Hilbert space. The *(operator) norm topology* is the topology on $B(H)$ induced by the operator norm. We define the *strong operator topology* as the coarsest topology on $B(H)$ such that for all $x \in H$, the maps

$$\begin{aligned} E_x : B(H) &\longrightarrow H \\ T &\longmapsto Tx \end{aligned}$$

are continuous. Moreover, we define the *weak operator topology* as the coarsest topology on $B(H)$ such that for all $x, y \in H$, the maps

$$\begin{aligned} E_{x,y} : B(H) &\longrightarrow \mathbb{C} \\ T &\longmapsto \langle x, Ty \rangle \end{aligned}$$

are continuous.

Definition 1.1.9 (group von Neumann algebra, [Kam19, Definition 2.23])**.** We define the *group von Neumann algebra* $\mathcal{R}G$ of G as the weak closure of the unital $*$-subalgebra $\rho(\mathbb{C}G)$ in $B(\ell^2 G)$.

Remark 1.1.10. Equivalently, we could also define the group von Neumann algebra to be the *strong* closure of $\rho(\mathbb{C}G)$.

Example 1.1.11 (finite groups)**.** Let G be a finite group of order $n \in \mathbb{N}$. Then $\ell^2 G = \mathbb{C}G$. Moreover, we have $B(\ell^2 G) \cong M_{n \times n}(\mathbb{C})$. In addition, the image of ρ is already closed. Thus, we have $\mathcal{R}G \cong \mathbb{C}G$ and in particular for the trivial group $\mathcal{R}1 \cong \mathbb{C}$.

Example 1.1.12 (the integers)**.** In [Kam19, Example 2.26], it is discussed that $\mathcal{R}\mathbb{Z} \cong L^\infty([-\pi, \pi]) \cong L^\infty(S^1)$, where the action of $\mathbb{C}[\mathbb{Z}]$ is given as follows: We identify $\mathbb{C}[\mathbb{Z}] \cong \mathbb{C}[z, z^{-1}]$ with the ring of Laurent polynomials. Then, the action of $\mathbb{C}[\mathbb{Z}]$ corresponds to the multiplication of functions in $L^\infty([-\pi, \pi])$ where $z = (x \mapsto e^{ix})$.

Recall that $L^\infty([-\pi, \pi])$ denotes the *essentially bounded* functions on the interval $[-\pi, \pi]$, i.e. the functions that are bounded up to a null set. We also identify two functions if they agree on a co-null set.

1.1.3 Traces and von Neumann Dimensions

We are now ready to define traces and dimension.

Definition 1.1.13 (trace)**.** We denote by $e \in \ell^2 G$ the unit element corresponding to the trivial element of the group. We define the *(von Neumann) trace* on $\mathcal{R}G$ by

$$\begin{aligned} \operatorname{tr}_{\mathcal{R}G} : \mathcal{R}G &\longrightarrow \mathbb{C} \\ T &\longmapsto \langle e, Te \rangle \end{aligned}$$

If the group G is clear from the context, we denote this trace simply by tr.

Remark 1.1.14 (elementary properties of the trace)*.* The trace is a $\mathbb{C}$-linear functional. Moreover, is satisfies the *trace property*: For all $S, T \in \mathcal{R}G$, we have $\operatorname{tr}_{\mathcal{R}G}(ST) = \operatorname{tr}_{\mathcal{R}G}(TS)$.

Remark 1.1.15 (trace of matrices)*.* We also obtain a trace on square matrices over $\mathcal{R}G$, namely by precomposition with the usual matrix trace $M_{n\times n}(\mathcal{R}G) \to \mathcal{R}G$, i.e. by taking the sum of the diagonal elements.

This trace is a linear functional satisfying the trace property and we still denote it by $\operatorname{tr}_{\mathcal{R}G}$ or tr.

Example 1.1.16 (trace on finite groups)**.** Let G be a finite group. As discussed in Example 1.1.11, we have $\mathcal{R}G = \mathbb{C}G$. Under this identification, the trace is given by

$$\begin{aligned} \operatorname{tr}_{\mathcal{R}G} : \mathbb{C}G &\longrightarrow \mathbb{C} \\ \sum_{g\in G} a_g \cdot g &\longmapsto a_e \end{aligned}$$

Example 1.1.17 (trace on $\mathbb{Z}$)**.** For $G = \mathbb{Z}$, recall that $\mathcal{R}G \cong L^\infty([-\pi,\pi])$ from Example 1.1.12. Under this identification, the trace is given by

$$\begin{aligned} \operatorname{tr}_{\mathcal{R}\mathbb{Z}} : L^\infty([-\pi,\pi]) &\longrightarrow \mathbb{C} \\ f &\longmapsto \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)\, dx \end{aligned}$$

Now, the idea to define the von Neumann dimension is the following: As in the classical case (where we consider, say, $\mathbb{C}$-vector spaces), the dimension is the trace of the identity map. However, this poses a problem whenever the module in question is not of type $(\ell^2 G)^n$ for some $n \in \mathbb{N}$.

In the case of a Hilbert $\mathcal{L}G$-module, we can still pass to this situation.

Definition 1.1.18 (Hilbert $\mathcal{L}G$-module, [Kam19, Definition 2.34])**.** A *Hilbert $\mathcal{L}G$-module* is a Hilbert space H with a linear isometric left G-action such that there exists a linear isometric G-embedding $H \hookrightarrow (\ell^2 G)^n$ for some $n \in \mathbb{N}$.

We can now define the von Neumann dimension.

Definition 1.1.19 (von Neumann dimension, [Kam19, Definition 2.37])**.** Let H be a Hilbert $\mathcal{L}G$-module. Choose a linear isometric G-embedding $H \hookrightarrow (\ell^2 G)^n$ for some $n \in \mathbb{N}$. We define the *von Neumann dimension* of H by

$$\dim_{\mathcal{R}G} H := \operatorname{tr}_{\mathcal{R}G}(p_{i(H)}) \in [0, \infty),$$

where $p_{i(H)} : (\ell^2 G)^n \to (\ell^2 G)^n$ is the orthogonal projection onto $i(H)$.

Remark 1.1.20 (independence of i)*.* The above definition of the von Neumann dimension is independent of the chosen embedding i. We can thus refer to the von Neumann dimension without having to specify this embedding i.

Example 1.1.21 (trivial group)**.** For the trivial group 1, we have $\mathcal{R}1 = \mathbb{C}$ (see Example 1.1.11) and Hilbert modules are just finite-dimensional inner product spaces. In particular, such a space is isomorphic to some $\mathbb{C}^n$ with the standard inner product. Since $\ell^2 1 = \mathbb{C}$, we can have the identity on $\mathbb{C}^n$ as the orthogonal projection for Definition 1.1.19 and we obtain

$$\dim_{\mathcal{R}1} \mathbb{C}^n = n = \dim_{\mathbb{C}} \mathbb{C}^n.$$

In particular, this means that the von Neumann dimension extends the classical notion of dimension of $\mathbb{C}$-vector spaces as the latter is recovered for $G = 1$.

Example 1.1.22 (finite groups)**.** Let G be a finite group and H be a Hilbert $\mathcal{L}G$-module. Then, H is of finite dimension as a $\mathbb{C}$-vector space. A calculation comparing with $\operatorname{tr}_{\mathcal{R}1}$ [Kam19, Theorem 2.36.iii], where 1 is the trivial subgroup in G, shows that

$$\dim_{\mathcal{R}G} H = \frac{\dim_{\mathbb{C}} H}{|G|}$$

Moreover, for all $k \in \mathbb{N}$, the we can turn $\mathbb{C}^k$ into a $\mathbb{C}G$-module by the trivial action. By writing $G = g_1, \dots, g_n$, we can embed $\mathbb{C}^k$ into $(\ell^2 G)^k$ by

$$\begin{aligned} \mathbb{C}^k &\longrightarrow (\ell^2 G)^k \\ e_i &\longmapsto \frac{1}{\sqrt{n}} \cdot (g_1 + \cdots + g_n) \cdot e_i \end{aligned}$$

Hence, $\mathbb{C}^k$ is a Hilbert $\mathcal{L}G$-module, and we obtain

$$\dim_{\mathcal{R}G} \mathbb{C}^k = \frac{k}{n}.$$

In particular, this shows that von Neumann dimensions do not necessarily have to be integer-valued.

Example 1.1.23 (the integers)**.** If $G = \mathbb{Z}$, for every measurable subset $A \subset [-\pi, \pi]$, the subspace $L^\infty A$ is closed in $L^\infty[-\pi, \pi]$. The orthogonal projection onto $L^\infty A$ is given by χ_A, the characteristic function of the set A. Thus, we have

$$\dim_{\mathcal{R}\mathbb{Z}} L^\infty A = \frac{1}{2\pi} \int_{-\pi}^{\pi} \chi_A(x)\, dx = \frac{\lambda(A)}{2\pi},$$

where $\lambda(A)$ is the Lebesgue measure of A. More generally, we obtain for all $k \in \mathbb{N}$ that

$$\dim_{\mathcal{R}\mathbb{Z}} (L^\infty A)^k = \frac{k \cdot \lambda(A)}{2\pi}.$$

In particular, this proves that the von Neumann dimension can be any nonnegative real number.

When working with the von Neumann-dimension, the following properties are often very useful.

Theorem 1.1.24 (Computing von Neumann-dimensions, [Kam19, Theorem 2.44])**.** *Let H, K and L be Hilbert $\mathcal{L}G$-modules.*

1. *Normalisation: We have* $\dim_{\mathcal{R}G}(\ell^2 G) = 1$.
2. *Faithfulness: We have* $\dim_{\mathcal{R}G}(H) = 0$ *if and only if* $H \cong 0$.
3. *Additivity: Let* $0 \longrightarrow H \xrightarrow{i} K \xrightarrow{p} L \longrightarrow 0$ *be a weakly exact sequence of Hilbert $\mathcal{L}G$-modules, i.e. i is injective,* $\ker p = \overline{\operatorname{im} i}$ *and* $\operatorname{im} p \subset L$ *is dense. Then, we have*
$$\dim_{\mathcal{R}G} K = \dim_{\mathcal{R}G} H + \dim_{\mathcal{R}G} L.$$
4. *Restriction: Let $G_0 \subset G$ be a subgroup of finite index. Then,*
$$\dim_{\mathcal{R}G_0} \operatorname{res}^G_{G_0}(H) = [G : G_0] \cdot \dim_{\mathcal{R}G} H.$$
Here, $\operatorname{res}^G_{G_0}(H)$ *denotes the Hilbert $\mathcal{L}G$-module H with its action restricted to G_0.*

1.2 ℓ^2-Betti Numbers

In this section, we introduce the notion of ℓ^2-Betti numbers. Like in the case of ordinary Betti numbers, these will be the dimensions of the homology groups. In order to be able to talk about these ℓ^2-homology groups, we need to introduce CW-complexes with a group action.

Let G be a (countable, discrete) group.

1.2.1 G-CW Complexes

Definition 1.2.1 ([Kam19, Definition 3.1]). Let X be a CW-complex. An action $G \curvearrowright X$ is *cellular* if for all $g \in G$ and open cells $E \subset X$, we have

1. The translate gE is again an open cell of X.
2. If $gE \cap E \neq \emptyset$, then g fixes all elements of E pointwise, i.e. for all $x \in E$, we have $gx = x$.

A *G-CW complex* is a CW complex with a cellular G-action.

The point of these two requirements is to ensure that we can describe a G-CW complex as a pushout as follows.

Proposition 1.2.2 (description as pushout, [Kam19, Theorem 3.2]). *Let X be a G-CW complex. Then, the skeleta $(X_n)_{n\in\mathbb{N}}$ of X are G-invariant subspaces and there exist pushouts in the category of G-spaces and G-maps*

$$\begin{array}{ccc} \coprod_{i\in I_n}(G/H_i)\times S^{n-1} & \longrightarrow & X_{n-1} \\ \downarrow & & \downarrow \\ \coprod_{i\in I_n}(G/H_i)\times D^{n} & \longrightarrow & X_{n} \end{array}$$

where $(H_i)_{i\in I_n}$ is a family of subgroups (we call these subgroups the stabiliser groups*), D^n is the closed disk of dimension n and the vertical maps are given by the canonical inclusions.*

For G-CW complexes, we call cells of the form $(G/H_i) \times D^n$ a *G-equivariant n-cell.*

Definition 1.2.3 (properties of G-CW complexes, [Kam19, Definition 3.3]). A G-CW complex X is called

- *(of) finite type* if it has finitely many G-equivariant n-cells for all $n \in \mathbb{N}$.
- *finite* if it has finitely many G-equivariant cells altogether.

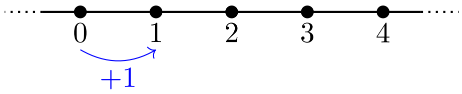

Figure 1.2: The real line $\mathbb{R}$ as a $\mathbb{Z}$-CW complex

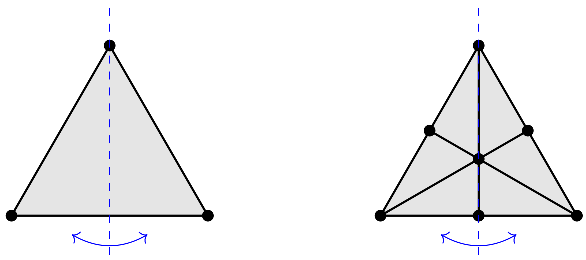

Figure 1.3: The 2-simplex and its barycentric subdivision: On the left, the reflection drawn in blue does not fix the 2-cell pointwise. On the right hand side, the 2-cells and their images are disjoint.

- *proper* if all stabiliser groups are finite.
- *free* if all stabiliser groups are trivial.

Example 1.2.4 (G-CW complexes)**.**

- The real line $\mathbb{R}$ is a $\mathbb{Z}$-CW complex as follows: As 0-cells, we have the subset $\mathbb{Z} \subset \mathbb{R}$ and for each two adjacent 0-cells, we glue in one 1-cell between them (see Figure 1.2). The action $\mathbb{Z} \curvearrowright \mathbb{R}$ given by translation is cellular. Thus, the real line $\mathbb{R}$ is a finite and free $\mathbb{Z}$-CW complex.
- The symmetric group S_3 acts on the 2-simplex. This action is *not* cellular as the non-trivial group elements fix the 2-cell altogether but not pointwise. However, if we pass to the barycentric subdivision, this action becomes cellular, and is finite and proper, but not free. This situation is depicted by Figure 1.3.
- Let X be a finite type, connected CW-complex and $p : Y \to X$ be a Galois covering of X. Let G denote the Deck transformation group of this covering p. Then, Y is a finite type G-CW complex.

  In particular, the universal covering $\widetilde{X}$ is a $\pi_1(X, x_0)$-CW complex of finite type. Moreover, it is free.

1.2.2 The ℓ^2-Completion of the Cellular Chain Complex

We will now introduce the ℓ^2-completion of the cellular chain complex. It is defined as follows.

Definition 1.2.5 (ℓ^2-completion, [Kam19, Definition 3.10])**.** Let X be a G-CW complex. The ℓ^2*-chain complex* of X is defined as the $\ell^2 G$-chain complex

$$C_*^{(2)}(X) := \ell^2 G \otimes_{\mathbb{Z}G} C_*(X),$$

where $C_*(X)$ is the cellular chain complex of X. The differentials of this chain complex are given by $d_*^{(2)} = \mathrm{id}_{\ell^2 G} \otimes d_*$.

Proposition 1.2.6 (functoriality and a combinatorial description, [Kam19, Theorem 3.11])**.** *The ℓ^2-chain complex defines a functor from the category of proper, finite type G-CW complexes to the category of chain complexes of Hilbert $\mathcal{L}(G)$-modules.*

Moreover, it has the following explicit description: If X is a proper, finite type G-CW complex, we choose a pushout as in Proposition 1.2.2. Then, for all $n \in \mathbb{N}$, there is a canonical isomorphism

$$C_n^{(2)}(X) \cong \bigoplus_{i \in I_n} \ell^2(G/H_i).$$

If, moreover, the chain complex is free, then it is of the form

$$\cdots \longrightarrow (\ell^2 G)^{k_2} \longrightarrow (\ell^2 G)^{k_1} \longrightarrow (\ell^2 G)^{k_0} \longrightarrow 0.$$

In addition, the differentials are given by right multiplication with matrices over $\mathbb{Z}G$. The matrices of the adjoints are obtained by transposing these matrices and applying the involution $g \mapsto g^{-1}$ to its elements.

1.2.3 ℓ^2-Betti Numbers

Now, we have all the tools at our hand to define ℓ^2-homology and ℓ^2-Betti numbers. This works as in the case of the ordinary Betti numbers: Homology measures the inexactness of a chain complex, and Betti numbers are the dimensions of homology groups.

Let X be a proper, finite type G-CW complex.

Definition 1.2.7 (ℓ^2-homology, [Kam19, Definition 3.12])**.** Let $n \in \mathbb{N}$. The *n-th (reduced) ℓ^2-homology* of X is the Hilbert $\mathcal{L}G$-module

$$H_n^{(2)}(G \curvearrowright X) := \ker d_n^{(2)} / \overline{\operatorname{im} d_{n+1}^{(2)}}.$$

Remark 1.2.8. Note that by taking the closure of $\operatorname{im} d_{n+1}^{(2)}$, we ensure that the quotient is again a Hilbert space (for details, see [Kam19, p. 11]).

Definition 1.2.9 (ℓ^2-Betti numbers,[Kam19, Definition 3.13])**.** Let $n \in \mathbb{N}$. The *n-th ℓ^2-Betti number* of X is defined by

$$b_n^{(2)}(G \curvearrowright X) := \dim_{\mathcal{R}G} H_n^{(2)}(G \curvearrowright X) \in [0, \infty).$$

Remark 1.2.10. If the group G and its action on X are clear from the context, we just write $b_n^{(2)}(X)$. However, it is important to be aware that this group action matters: Different group actions lead to different ℓ^2-Betti numbers in general.

Note that this number is a priori just a nonnegative real number. Versions of Atiyah's conjecture state that these values are always integral or rational (see Question 1.3.1 and Conjecture 1.3.5). The question which values can occur – given certain assumptions on G and X – will pervade this thesis.

Remark 1.2.11. The ℓ^2-Betti numbers are an extension of ordinary Betti numbers: If X is a CW-complex, the trivial action of the trivial group 1 on X is cellular. Using Example 1.1.21, we see that in this case, $b_n^{(2)}(1 \curvearrowright X)$ coincides with the ordinary Betti number of X (with coefficients in $\mathbb{C}$).

Example 1.2.12. If X is a finite type CW-complex and $x_0 \in X$, we obtain an action of $\pi_1(X, x_0)$ on its universal covering $\widetilde{X}$. By Example 1.2.4, this action is cellular. We can therefore consider the ℓ^2-Betti numbers

$$b_n^{(2)}(\widetilde{X}) := b_n^{(2)}(\pi_1(X, x_0) \curvearrowright \widetilde{X}).$$

In the literature (and also in the introduction to this thesis), sometimes, one finds the notation $b_n^{(2)}(X)$. However, this might cause confusion with ℓ^2-Betti numbers of X with respect to a cellular action of a group that is clear from the context. In this thesis, we will therefore try to avoid this confusion by writing $b_n^{(2)}(\widetilde{X})$.

The following theorem is quite useful for calculating ℓ^2-Betti numbers.

Theorem 1.2.13 (Computing ℓ^2-Betti numbers, [Kam19, Theorem 3.18])**.**

1. *Homotopy invariance: Let $f : X \to Y$ be a G-homotopy equivalence of proper, finite type G-CW-complexes, i.e. a cellular G-map that is a homotopy equivalence and such that all the occurring homotopies are cellular G-maps. Then, for all $n \in \mathbb{N}$, we have $b_n^{(2)}(X) = b_n^{(2)}(Y)$.*
2. *Zeroth ℓ^2-Betti number: Let X be a non-empty, finite type, proper, connected G-CW complex. Then, $b_0^{(2)}(X) = \frac{1}{|G|}$. For the case that G is infinite, we set $\frac{1}{\infty} := 0$.*

3. *Künneth formula: Let X_1 be a proper, finite type G_1-CW complex and X_2 be a proper, finite type G_2-CW complex. Then, $X_1 \times X_2$ is a proper, finite type $G_1 \times G_2$-CW complex (w.r.t. the canonical action) and we have for all $n \in \mathbb{N}$*

$$b_n^{(2)}(G_1 \times G_2 \curvearrowright X_1 \times X_2) = \sum_{p+q=n} b_p^{(2)}(G_1 \curvearrowright X_1) \times b_q^{(2)}(G_2 \curvearrowright X_2).$$

4. *Restriction: Let X be a proper, finite type G-CW complex and let $G_0 \subset G$ be a subgroup of finite index. Then, by $\operatorname{res}_{G_0}^G X$, we denote the same CW-complex with the action restricted to G_0. Then, $\operatorname{res}_{G_0}^G X$ is a proper, finite type G_0-CW complex, and for all $n \in \mathbb{N}$, we have*

$$b_n^{(2)}(\operatorname{res}_{G_0}^G X) = [G : G_0] \cdot b_n^{(2)}(G \curvearrowright X).$$

Example 1.2.14 (finite groups)**.** Let G be a finite group and X be a finite type G-CW complex. Then, X is proper. Because G has the trivial group as a subgroup of index G, the above Theorem 1.2.13 yields that

$$b_n^{(2)}(G \curvearrowright X) = \frac{1}{|G|} \cdot b_n(X).$$

In particular, this shows that for finite groups, the theory essentially coincides with the theory of ordinary Betti numbers. Thus, we will focus our attention to ℓ^2-Betti numbers of CW-complexes over infinite groups.

Example 1.2.15 (the real line)**.** We return to our example of the real line $\mathbb{R}$ as a $\mathbb{Z}$-CW complex (see Example 1.2.4). For all $n \in \mathbb{N}$, we have

$$b_n^{(2)}(\mathbb{Z} \curvearrowright \mathbb{R}) = 0.$$

One can see this either by an explicit computation of the corresponding chain complex (see [Kam19, Example 3.15]) or by the following argument: We consider the subgroup $2\mathbb{Z} \subset \mathbb{Z}$. Note that $2\mathbb{Z} \cong \mathbb{Z}$ and that the action $2\mathbb{Z} \curvearrowright \mathbb{R}$ is $\mathbb{Z}$-homotopy equivalent to the action $\mathbb{Z} \curvearrowright \mathbb{R}$. Thus, by the above Theorem 1.2.13, for all $n \in \mathbb{N}$, we have

$$\begin{aligned} b_n^{(2)}(\mathbb{Z} \curvearrowright \mathbb{R}) &= [\mathbb{Z} : 2\mathbb{Z}] \cdot b_n^{(2)}(2\mathbb{Z} \curvearrowright \mathbb{R}) \\ &= [\mathbb{Z} : 2\mathbb{Z}] \cdot b_n^{(2)}(\mathbb{Z} \curvearrowright \mathbb{R}). \end{aligned}$$

From $[\mathbb{Z} : 2\mathbb{Z}] = 2$, we thus obtain $b_n^{(2)}(\mathbb{Z} \curvearrowright \mathbb{R}) = 0$.

We finish this section with the following theorem which allows to calculate the Euler characteristic via ℓ^2-Betti numbers.

Proposition 1.2.16 (Euler-Poincaré formula, [Kam19, Theorem 3.19])**.** *Let X be a finite CW-complex and $x_0 \in X$. Then,*

$$\chi(X) = \sum_{n\in\mathbb{N}} (-1)^n \cdot b_n^{(2)}(\pi_1(X,x_0) \curvearrowright \widetilde{X}).$$

1.2.4 The Set of ℓ^2-Betti Numbers

One important question in the theory of ℓ^2-Betti numbers is the following: Which values can occur as ℓ^2-Betti numbers?

It turns out that we can view this question from both a topologic and an algebraic viewpoint. We start with the following definitions.

Definition 1.2.17 (the topological viewpoint)**.** Let G be a finitely generated (discrete) group. We define the *set of (topological) ℓ^2-Betti numbers arising from* G as

$$B^{(2)}(G) := \{b_n^{(2)}(G \curvearrowright X) \mid n \in \mathbb{N}, X \text{ finite, free, connected } G\text{-CW complex}\}.$$

and additionally, if G is finitely presented,

$$\begin{aligned}\widetilde{B}^{(2)}(G) &:= \{b_n^{(2)}(G \curvearrowright X) \mid n \in \mathbb{N}, X \text{ finite, free, \textit{simply connected} } G\text{-CW complex}\}\\ &= \{b_n^{(2)}(\widetilde{X}) \mid n \in \mathbb{N}, X \text{ finite, connected CW-complex with } \pi_1(X,x_0) \cong G\}.\end{aligned}$$

where the last equality is due to Example 1.2.4. Quantifying over all finitely generated (resp. finitely presented) groups, we set

$$B^{(2)} = \bigcup_{G \text{ fin. gen.}} B^{(2)}(G) \quad \text{and} \quad \widetilde{B}^{(2)} = \bigcup_{G \text{ fin. pres.}} \widetilde{B}^{(2)}(G).$$

Moreover, there is the algebraic viewpoint that just talks about matrices over the group ring.

Definition 1.2.18 (the algebraic viewpoint)**.** Let $\mathbb{Z} \subset R \subset \mathbb{C}$ be a ring and G be a finitely generated (discrete) group. We define the *set of algebraic ℓ^2-Betti numbers arising from* G (with *coefficients* in R) as

$$B_R^{(2)}(G) := \{\dim_{\mathcal{R}G} \ker(\cdot A) \mid A \in M_{k\times l}(RG), k, l \in \mathbb{N}\}.$$

Remark 1.2.19. In the above definition, we can also assume these matrices A to be square and self-adjoint: For a general $A \in M_{k\times l}(RG)$, we have that AA^* is square and self-adjoint. Moreover, $\ker(\cdot AA^*) = \ker(\cdot A)$ because the following holds:

If $x \in \ker(\cdot A)$, then clearly $x \in \ker(\cdot AA^*)$. Conversely, if $x \in \ker(\cdot AA^*)$, then $(\cdot AA^*)x = (\cdot A^*) \circ (\cdot A)x = 0$, hence

$$\begin{aligned}\langle(\cdot A)x, (\cdot A)x\rangle &= \langle x, (\cdot A^*) \circ (\cdot A)x\rangle \\ &= 0\end{aligned}$$

and hence (by positive definiteness $(\cdot A)x = 0$, i.e. $x \in \ker(\cdot A)$.

As it turns out, these sets of ℓ^2-Betti numbers actually coincide.

Theorem 1.2.20. *Let G be a finitely generated group. Then, we have*

$$B^{(2)}(G) = B^{(2)}_{\mathbb{Z}}(G).$$

If moreover, G is finitely presented, we even have

$$B^{(2)}(G) = B^{(2)}_{\mathbb{Z}}(G) = \widetilde{B}^{(2)}(G).$$

Proof. First, let $r \in B^{(2)}(G)$, i.e. there exist $n \in \mathbb{N}$ and a finite, free, connected G-CW complex X such that $r = b_n^{(2)}(G \curvearrowright X)$. Since X is free, its ℓ^2-chain complex around n has the following shape

$$(\ell^2 G)^{k_{n+1}} \xrightarrow{d_{n+1}^{(2)}} (\ell^2 G)^{k_n} \xrightarrow{d_n^{(2)}} (\ell^2 G)^{k_{n-1}}.$$

By definition,

$$r = b_n^{(2)}(G \curvearrowright X) = \dim_{\mathcal{R}G} \ker d_n / \overline{\operatorname{im} d_{n+1}}.$$

By the ℓ^2-Hodge-de-Rham decomposition [Kam19, Proposition 3.23], we have

$$\ker d_n / \overline{\operatorname{im} d_{n+1}} \cong \ker \Delta_n,$$

where $\Delta_n := d_n^{(2)*} d_n^{(2)} + d_{n+1}^{(2)} d_{n+1}^{(2)*}$ is called the *n-th ℓ^2-Laplacian.* Hence, we have $r = \dim_{\mathcal{R}G} \ker \Delta_n$, where Δ_n is given by right-multiplication with some $n_k \times n_k$-matrix over $\mathbb{Z}G$.

The converse inclusion is given by the following Proposition 1.2.21. □

Proposition 1.2.21 (realising matrices, [Kam19, Proposition 3.29])**.** *Let G be a finitely generated group that is generated by r elements, and $A \in M_{k\times l}(\mathbb{Z}G)$. Then, there exists a free G-CW complex X consisting of k equivariant 3-cells, l equivariant 2-cells, r equivariant 1-cells and one equivariant 1-cell such that the third ℓ^2-differential*

$$d_3^{(2)} : C_3^{(2)}(X) \longrightarrow C_2^{(2)}(X)$$

can be identified with the right multiplication operator $(\ell^2 G)^k \xrightarrow{\cdot A} (\ell^2 G)^l$ and hence $b_3^{(2)}(G \curvearrowright X) = \dim_{\mathcal{R}G} \ker(\cdot A)$.

If G is finitely presented, we can additionally obtain such a G-CW complex X that is simply connected.

Remark 1.2.22. This proposition can be proved by an explicit construction. For details, see [Kam19, Proposition 3.29].

We end this section by stating some basic facts about the structure of the set of ℓ^2-Betti numbers of a group.

Proposition 1.2.23 (basic facts about $B^{(2)}(G)$)**.** *Let G be a finitely generated group. Then, the following hold:*

1. *We have $\mathbb{N} \subseteq B^{(2)}(G)$.*
2. *The set $B^{(2)}(G)$ is closed under sums.*
3. *Monotonicity: If $H \subseteq G$ is a subgroup, then $B^{(2)}(H) \subseteq B^{(2)}(G)$.*
4. *If G_1 and G_2 are finitely generated groups, $a_1 \in B^{(2)}(G_1)$ and $a_2 \in B^{(2)}(G_2)$, we have*
$$a_1 + a_2 \in B^{(2)}(G_1 \times G_2)$$
and
$$a_1 \cdot a_2 \in B^{(2)}(G_1 \times G_2).$$
5. *If $a \in B^{(2)}(G)$ and $q \in \mathbb{Q}_{\geq 0}$, there exists a finitely generated group H such that $q \cdot a \in B^{(2)}(H)$.*

Proof. The first statement can be shown using matrices of the appropriate size filled with zeros. Statement 2 follows by taking block matrices. Statement 3 follows from the fact that $\mathbb{C}H \hookrightarrow \mathbb{C}G$ is a trace-preserving $*$-homomorphism (see [Gra14, Lemma 6.1]). For the fourth part, see [PSZ15, Lemma 11.2] and [PSZ15, Lemma 11.3]. Finally, for the fifth claim, see [Gra14, Lemma 6.2]. □

1.2.5 ℓ^2-Betti Numbers of Groups

Finally, we will define ℓ^2-Betti numbers *of* groups. These are *not* to be confused with the ℓ^2-Betti numbers *arising from* a group, which is generally a much larger set.

We use a general trick to transfer notions on topological spaces that are homotopy invariant to groups: We construct the *classifying space* for a group and then define the ℓ^2-Betti numbers of a group to be the ℓ^2-Betti numbers of its classifying space.

Proposition 1.2.24 (classifying space,[Kam19, Theorem 4.2])**.** *Let G be a group. Then, there exists a free, connected G-CW complex X such that X is weakly contractible, i.e. for all $n \geq 1$, we have $\pi_n(X) \cong 1$.*

Moreover, this G-CW complex X is unique up to G-homotopy equivalence.

Definition 1.2.25 (classifying space)**.** In the situation of the proposition above, we call such G-CW complexes a *(model for a) classifying space* for G and denote one choice of such a model by EG.

If $n \in \mathbb{N}$, we say that a group G is *of type F_n* if there exists a classifying space EG for G with finite n-skeleton.

Example 1.2.26.

- Every group is of type F_0.
- A group is of type F_1 if and only if it is finitely generated.
- A group is of type F_2 if and only if it is finitely presented.

We can now define ℓ^2-Betti numbers of groups.

Definition 1.2.27 (ℓ^2-Betti numbers of groups)**.** Let $n \in \mathbb{N}$ and G be a group of type F_{n+1}. Denote by EG a model for the classifying space of G with finite $(n+1)$-skeleton EG_{n+1}. Then, we define the *n-th ℓ^2-Betti number of* G by

$$b_n^{(2)}(G) := b_n^{(2)}(G \curvearrowright EG_{n+1}) \in [0, \infty).$$

Remark 1.2.28. It is possible to generalise this definition also in the case that EG is not necessarily of type F_{n+1}. Using this approach, we would no longer need to view EG_{n+1} but could instead take the ℓ^2-Betti numbers of EG (which coincide up to degree n).

However, in this generalisation, also the value $+\infty$ can occur. Since this would need more theory yet we will not use this notion later on, this generalisation will not be treated here. Details can be found in the book by Kammeyer [Kam19, Chapters 4.2, 4.3].

In analogy to Theorem 1.2.13, there is also the following theorem to compute ℓ^2-Betti numbers of groups.

Theorem 1.2.29 (Computing ℓ^2-Betti numbers of groups, [Kam19, Theorem 4.15])**.** *Let $n \in \mathbb{N}$ and G, G_1, G_2 be groups of finite type F_{n+1}. Then, the following hold:*

- *Zeroth ℓ^2-Betti number: $b_0^{(2)}(G) = \frac{1}{|G|}$ (where again $\frac{1}{\infty} := 0$).*
- *Künneth formula: We have*

$$b_n^{(2)}(G_1 \times G_2) = \sum_{p+q=n} b_p^{(2)}(G_1) \cdot b_q^{(2)}(G_2).$$

- *For $n \geq 2$, we have*

$$b_n^{(2)}(G_1 * G_2) = b_n^{(2)}(G_1) + b_n^{(2)}(G_2)$$
$$b_1^{(2)}(G_1 * G_2) = 1 + b_1^{(2)}(G_1) - \frac{1}{|G_1|} + b_1^{(2)}(G_2) - \frac{1}{|G_2|}.$$

- *Restriction: If $G_0 \subset G$ is a subgroup of finite index, then*

$$b_n^{(2)}(G_0) = [G : G_0] \cdot b_n^{(2)}(G).$$

We finish this section by giving a few examples of ℓ^2-Betti numbers of groups. All examples (with calculations) can be found in the book by Kammeyer [Kam19, Table 4.1].

Example 1.2.30.

- $b_n^{(2)}(\mathbb{Z}^k) = 0$ for all $k, n \in \mathbb{N}$.
- Let $k \in \mathbb{N}_{>0}$ and F_k denote the free group on k generators. Then, $b_1^{(2)}(F_k) = k - 1$ and $b_n^{(2)}(F_k) = 0$ for $n \neq 1$.
- Let G be a finite group. Then, $b_0^{(2)}(G) = \frac{1}{|G|}$ and $b_n^{(2)}(G) = 0$ for $n > 0$.

1.3 Further Questions about ℓ^2-Betti Numbers

In this section, we will treat two interesting aspects of ℓ^2-Betti numbers, namely Atiyah's conjecture and Lück's Approximation Theorem.

1.3.1 Atiyah's Conjecture

In the definition of ℓ^2-Betti numbers, a priori, the resulting numbers are non-negative real numbers. However, in many examples, one finds that these numbers are actually rational. Thus, Atiyah asked the following natural question.

Question 1.3.1 (Atiyah's question (reformulated), [Ati76, p. 72]). Does $B^{(2)}$, the set of ℓ^2-Betti numbers arising from all finitely generated groups (see Definition 1.2.17), contain irrational numbers?

This question was finally answered more than 30 years later by Tim Austin.

Theorem 1.3.2 ([Aus13, Theorem 1.1, Corollary 1.2]). *The set $B^{(2)}$ is uncountable. In particular, it contains irrational and even transcendental elements.*

The irrationality of some ℓ^2-Betti numbers was independently also shown by Lehner and Wagner [LW13, Theorem 2.3]. Some years later, even the following was shown independently by Grabowski [Gra14, Theorem 1.3] as well as Pichot, Schick and Zuk [PSZ15, Theorem 11.1].

Theorem 1.3.3. *We have $B^{(2)} = \mathbb{R}_{\geq 0}$.*

In order to prove this theorem, Grabowski and well as Pichot-Schick-Zuk construct groups that contain the *wreath product* $(\mathbb{Z}/2\mathbb{Z}) \wr \mathbb{Z}$, i.e. the semidirect product

$$\bigoplus_{\mathbb{Z}} (\mathbb{Z}/2\mathbb{Z}) \rtimes \mathbb{Z}$$

(where $1 \in \mathbb{Z}$ acts by shifting) as a subgroup. This group is called the *lamplighter group*. It is an open question whether the converse is also true, more precisely:

Question 1.3.4 (Grabowski, [Gra16, Problem 1.3]). Let G be a finitely generated group such that $B^{(2)}(G)$ contains an irrational number. Does there exist a prime p such that $(\mathbb{Z}/p\mathbb{Z}) \wr \mathbb{Z}$ is a subgroup of G?

Moreover, there is another version of the Atiyah conjecture, sometimes called the *strong Atiyah conjecture*.

Conjecture 1.3.5 ((strong) Atiyah conjecture, [Kam19, Conjecture 3.30])**.** *Let* $\mathbb{Z} \subset R \subset \mathbb{C}$ *be a ring and* G *be a finitely generated group whose finite subgroups are of bounded order. Let* $\operatorname{lcm}(G)$ *be the least common multiple of the occurring orders. Then,* $B_R^{(2)}(G) \subset \frac{1}{\operatorname{lcm}(G)} \cdot \mathbb{N}$.

The importance of this conjecture is due to its relation to the Kaplansky conjecture.

Theorem 1.3.6 ([Kam19, Theorem 3.32])**.** *Let* G *be a finitely generated, torsion-free group that satisfies the strong Atiyah conjecture with coefficients in a ring* $R \subset \mathbb{C}$*. Then,* RG *satisfies the* Kaplansky conjecture*, i.e.* RG *contains no non-trivial zero-divisors.*

For a large class of groups, the strong Atiyah conjecture was shown by P. Linnel.

Theorem 1.3.7 (Linnel, [Kam19, Theorem 3.33])**.** *Let* G *be a group belonging to the class* $\mathcal{C}$ *(see below, Definition 1.3.8). Assume that there is a bound on the orders of finite subgroups of* G*. Then,* G *satisfies the strong Atiyah conjecture with* $R = \mathbb{C}$*.*

The class $\mathcal{C}$ is defined as follows.

Definition 1.3.8. We define the class of *elementary amenable* groups, denoted $\mathcal{E}$, to be the smallest class of groups that contains all finite groups, all abelian groups, and is closed under taking subgroups, quotients, extensions and directed unions.

We then define $\mathcal{C}$ to be the smallest class of groups that is closed under directed unions, contains all free groups and all groups G that occur in an extension of the form

$$1 \longrightarrow N \longrightarrow G \longrightarrow A \longrightarrow 1$$

with $N \in \mathcal{C}$ and $A \in \mathcal{E}$.

1.3.2 Spectral Measures and Lück's Approximation Theorem

Now, we will take a look at Lück's approximation theorem. It sets into relation the ℓ^2-Betti numbers with the ordinary Betti numbers. It was originally proved by W. Lück in 1994 [Lüc94, Theorem 0.1].

Theorem 1.3.9 (Lück's approximation theorem, [Kam19, Theorem 5.2])**.** *Let* X *be a free, finite type* G*-CW complex. Suppose that* G *is residually finite, and let* $(G_i)_{i\in\mathbb{N}}$ *be a residual chain of* G*. Then, for every* $n \in \mathbb{N}$*, we have*

$$b_n^{(2)}(G \curvearrowright X) = \lim_{i\to\infty} \frac{b_n(G_i \backslash X)}{[G : G_i]}.$$

Here, b_n *denotes the (ordinary)* n*-th Betti number of a CW complex.*

First recall the definition of a residual chain.

Definition 1.3.10 (residual chain, residual finiteness)**.** Let G be a group. A *residual chain* is a sequence $(G_i)_{i\in\mathbb{N}}$ of finite index normal subgroups in G that are nested, i.e.

$$G = G_0 \supset G_1 \supset G_2 \supset \dots$$

and such that $\bigcap_{i\in\mathbb{N}} G_i = \{e\}$. We call a group *residually finite* if such a residual chain exists.

Note that for a countable (discrete) group G, a residual chain exists if and only if G is residually finite.

We will now sketch the first part of the proof of the Lück approximation theorem. We will follow the proof in the book of Kammeyer [Kam19, Chapter 5]. The first observation is the following:

Remark 1.3.11. In the situation of the Lück approximation theorem (Theorem 1.3.9), for every $i \in \mathbb{N}$, we have a canonical action $G/G_i \curvearrowright G_i\backslash X$. Note that G/G_i is a finite group of cardinality $[G : G_i]$. Hence, by Example 1.2.14, we can rewrite the claim of the Lück approximation theorem in the following form: For all $n \in \mathbb{N}$, we have

$$b_n^{(2)}(G \curvearrowright X) = \lim_{i\to\infty} b_n^{(2)}(G/G_i \curvearrowright G_i\backslash X).$$

For the proof of this reformulated version, one uses functional calculus to transform this claim into a measure-theoretic question. Here, we sketch the most important results, all details can be found in the book of Kammeyer [Kam19, Chapter 5.2].

Definition 1.3.12 (spectrum)**.** Let $T \in B(H)$ be a bounded operator on a separable Hilbert space H. Then, its *spectrum* is defined as

$$\sigma(T) := \{\zeta \in \mathbb{C} \mid \zeta \cdot \mathrm{id}_H - T \text{ is not bijective}\}$$

If T is self-adjoint, we have $\sigma(T) \subset \mathbb{R}$. The following theorem says that we can apply continuous functions to bounded operators.

Theorem 1.3.13 (Continuous functional calculus, [Kam19, Theorem 5.6])**.** *Let $T \in B(H)$ be self-adjoint. Then, there is a unique isometric $*$-embedding of C^*-algebras*

$$\begin{aligned} C(\sigma(T), \mathbb{C}) &\longrightarrow B(H) \\ f &\longmapsto f(T) \end{aligned}$$

such that for all polynomials $p \in \mathbb{C}[x]$, we have that $p(T)$ is the evaluation of p on T.

We can use this to define the *spectral measure*.

Definition 1.3.14. Let $T \in B(H)$ be self-adjoint. Then, the *spectral measure* of T associated with x is the unique measure $\mu_{x,T}$ such that for all $f \in C(\sigma(T), \mathbb{C})$, we have

$$\langle x, f(T)x \rangle = \int f \, d\mu_{x,T}$$

The existence and uniqueness of this measure is guaranteed essentially by the Riesz Representation Theorem.

From this point on, we skip some details and pass directly to the application to our situation.

Definition 1.3.15 (spectral measure, $\ell^2 G$-version)**.** Let G be a countable, discrete group. Let $n \in \mathbb{N}$ and $A \in M_{n\times n}(\mathbb{Z}G)$ be self-adjoint. Then, the *spectral measure* of A is the spectral measure $\mu_{\epsilon,\cdot A}$ of $\cdot A : (\ell^2 G)^n \to (\ell^2 G)^n$ associated with $\epsilon = (e, \ldots, e) \in (\ell^2 G)^n$, where $e \in \ell^2 G$ is the characteristic function of the neutral element in G. We denote this spectral measure by μ_A.

We can view the spectral measure as a measure on the interval $[0, ||\cdot A||]$ (with the Borel σ-algebra). It is characterised by the following property.

Proposition 1.3.16 (characterisation of the spectral measure)**.** *Let G be a countable, discrete group, $n \in \mathbb{N}$ and $A \in M_{n\times n}(\mathbb{Z}G)$ be self-adjoint. Then, the spectral measure is the unique measure μ_A on the interval $[0, ||\cdot A||]$ such that: For all polynomials $p \in \mathbb{R}[x]$, we have*

$$\operatorname{tr}_{\mathcal{R}G}(p(A)) = \int_0^{||\cdot A||} p(x) \, d\mu_A(x).$$

We can also express the dimension of the kernel via the spectral measure.

Proposition 1.3.17 (kernel via spectral measure)**.** *Let G be a countable, discrete group, $n \in \mathbb{N}$ and $A \in M_{n\times n}(\mathbb{Z}G)$ be self-adjoint. Then, we have*

$$\dim_{\mathcal{R}G} \ker(\cdot A) = \mu_A(\{0\}).$$

We can then use these two proposition to rewrite the claim of the Lück approximation theorem. We also pass to the algebraic viewpoint and obtain the following.

Theorem 1.3.18 (Lück's approximation theorem, measure theoretic version)**.** *Let X be a free, finite type G-CW complex. Suppose that G is residually finite, and $(G_i)_{i\in\mathbb{N}}$ be a residual chain of G. Let $n \in \mathbb{N}$ and let $\Delta_n \in M_{k\times k}(\mathbb{Z}G)$ denote the n-th Laplacian matrix of the ℓ^2-chain complex of $G \curvearrowright X$. Moreover, for $i \in \mathbb{N}$, let $\pi_i(\Delta_n) \in M_{k\times k}(\mathbb{Z}(G/G_i))$ be the entrywise projection of Δ_n to $\mathbb{Z}(G/G_i)$. Then,*

$\pi_i(\Delta_n)$ is the n-th Laplacian of the ℓ^2-chain complex of $G/G_i \curvearrowright G_i\backslash X$. Moreover, we have

$$\mu_{\Delta_n}(\{0\}) = \lim_{i\to\infty} \mu_{\pi_i(\Delta_n)}(\{0\}).$$

We will not prove this theorem here. For a proof, see Lück's original article [Lüc94] or Kammeyer's book [Kam19, Chapter 5.3]. Moreover, in this thesis, we will later prove a version of this statement that quantifies the rate of convergence (see Theorem 3.4.5).

2 Introduction to Computability

2.1 A Naive Introduction into Computability

2.1.1 Algorithms

In this section, we introduce basic notions of algorithms and computability. However, we will not make precise what we mean by 'algorithm'. There exist precise formulations of what an algorithm is, for instance a Turing machine or an unlimited register machine. An introduction to these and various other approaches, which by Church's Thesis are equivalent, can be found in the book by Cutland [Cut80, Chapters 1 and 3] and Fernández [Fer09]. However, many of these approaches have the disadvantage that it is rather inconvenient to precisely prove computability of more complicated functions.

Regarding basic notions of algorithms, we will mainly follow Cormen et. al. [Cor+09, pp. 5–6].

Remark 2.1.1 (algorithms). We think of an *algorithm* as a finite, well-defined computational procedure. It takes some (finite set of) values as input, then applies a sequence of computational steps to this input. This sequence is either finite and in the last step, the algorithm outputs (a finite set of) values, or this sequence is infinite. In the former case, we say that the algorithm *halts* or *terminates*.

We can specify an algorithm by a clear description in English (or any other language) of what computational steps to apply. Alternatively (and often less ambiguously), we can state such a description in pseudocode that should be precise enough for our needs. We largely follow the conventions by Cormen et. al. [Cor+09, pp. 20–22] for writing pseudocode, which should be well-understandable by people familiar with programming languages like C, C++, Python, Java, etc.

Note that we will be rather vague on what level of precision the description of an algorithm has to satisfy. The only requirement is that an algorithm must give a relatively precise description of the computational procedure to be followed. If we formalised algorithms, say, via Turing machines, in theory, we would need to be able to produce the description of a Turing machine for each algorithm that we state.

We will make this more intuitive by stating some (non-)examples.

Example 2.1.2 (algorithms)**.** We accept the following descriptions as algorithms:

```
Input  : n ∈ ℕ
Return n
```

Algorithm 1: An easy example

```
Input  : n ∈ ℕ
Set m := n + 1
while true do
    Replace m by m^2 + m − n
    if m ≤ 42 then
        Return m
    end
end
```

Algorithm 2: Another example

'On input $n \in \mathbb{N}$, output n' counts as an algorithm: If we have n as an input, it is sensible that we can use it as an output. In most formalisations, it is easy to make this precise. One pseudocode formalisation is given by Algorithm 1.

The pseudocode given by Algorithm 2 defines an algorithm since the instructions are clear. It is however not obvious whether or not this algorithm terminates on any input and which function it computes.

Example 2.1.3 (no algorithms)**.** We do not accept the following descriptions as algorithms:

- 'On input $n \in \mathbb{N}$, output any natural number' is *not* an algorithm: It is not clear how such a natural number should be chosen.
- 'On input $n \in \mathbb{N}$, output the n-th digit of π (in decimal representation)' is *not* precise enough: It is unclear, how one can calculate the n-th digit of π. Such an algorithm exists, however, but to define it, we need a precise description of what is to be done.
- 'On input $n \in \mathbb{N}$, output *yes* if n is prime, otherwise *no*' does not count either as an algorithm in the precise sense. However, it is not hard to write down a more precise description of how to check for primality.

2.1.2 Computability of Functions

If A is an algorithm and x an input to A, we denote by $A(x)$ the sequence of computational steps that A performs on input x. If this sequence is finite, then we denote also by $A(x)$ the output that A gives on input x.

Definition 2.1.4 (computable function)**.** A partial function $f : \mathbb{N}^k \to \mathbb{N}^l$ is called *computable* if there exists an algorithm A such that for all $x \in \mathbb{N}^k$:

- If $x \in \operatorname{dom}(f)$, then $A(x)$ halts and outputs the value $f(x)$.
- If $x \notin \operatorname{dom}(f)$, then $A(x)$ does not halt.

Remark 2.1.5. More generally, we can call functions $f : X \to Y$ *computable* if for suitable codings $X \cong \mathbb{N}^k, Y \cong \mathbb{N}^l$, the composed map $\mathbb{N}^k \to \mathbb{N}^l$ is computable in the above sense.

One special case of this is the case of rational values: For instance, we can say that a function $f : \mathbb{N}^k \to \mathbb{Q}$ is computable, if there exist computable functions $g_1, g_2 : \mathbb{N}^k \to \mathbb{N}$ such that $\operatorname{dom} f = \operatorname{dom} g_1 = \operatorname{dom} g_2$, and for all $x \in \operatorname{dom} f$, we have $f(x) = \frac{g_1(x)}{g_2(x)}$.

Lemma 2.1.6 (inheritance properties)**.** *Sums, products, quotients and compositions of computable functions are again computable.*

Proof (Sketch). For all of these operations, we can give algorithms relative to algorithms that compute the summands (resp. factors, etc.). □

Example 2.1.7. The following functions are computable:

- Constant functions
- projections to factors and inclusions
- $n \mapsto p(n)$, where $p \in \mathbb{Q}[x]$ is a polynomial with rational coefficients.
- $n \mapsto n!$
- $n \mapsto n$-th prime number (ordered by size)
- $n \mapsto n$-digit in the decimal expansion of π
- (A, B matrices over $\mathbb{Z}$)$\mapsto A \cdot B$ (when $A \cdot B$ is defined).
- $n \mapsto \lceil \log n \rceil$ and $n \mapsto \lfloor \log n \rfloor$. For details on the computation of logarithms, see an article by Egbert [Egb78].

2.1.3 Decidability

We will now turn to the decidability of problems.

Definition 2.1.8 ((semi-)decidable sets, [Cut80, Definition 7.1.1 and 7.1.2])**.** A subset $X \subset \mathbb{N}^k$ is called *decidable* (or *computable* or *recursive*) if the function

$$\begin{aligned} \chi_X : \mathbb{N}^k &\longrightarrow \mathbb{N} \\ x &\longmapsto \begin{cases} 1 & \text{if } x \in X \\ 0 & \text{if } x \notin X \end{cases} \end{aligned}$$

is computable. We say that an algorithm which computes χ_X, *decides* X.

We call X *semi-decidable* or *recursively enumerable* if the function

$$\begin{aligned} \mathbb{N}^k &\longrightarrow \mathbb{N} \\ x &\longmapsto \begin{cases} 1 & \text{if } x \in X \\ \text{undefined} & \text{if } x \notin X \end{cases} \end{aligned}$$

is computable. Here, the above notation denotes the partial function $\mathbb{N}^k \to \mathbb{N}$, whose domain is $X \subset \mathbb{N}^k$, and on which the function is constant 1.

Remark 2.1.9. For sets that are not a subset of some $\mathbb{N}^k$, we compose with a suitable coding (as described in Remark 2.1.5) to obtain a notion of decidability.

Example 2.1.10 ((Semi-)decidable sets)**.**

- All finite sets are decidable. (We can write the given finite set into the algorithm, then the algorithm checks if the input is equal to one of those elements.)
- The subset of $\mathbb{N}$ given by the even numbers is decidable.
- The subset of $\mathbb{N}$ given by the prime numbers is decidable.
- The set
$$\{(x, y) \in \mathbb{N}^2 \mid x \leq y\}$$
is decidable.
- Decidable sets are semi-decidable.
- A subset $X \subset \mathbb{N}^k$ is decidable if and only if both X and $\mathbb{N}^k \backslash X$ are semi-decidable. [Cut80, Theorem 7.2.6]
- Choose an enumeration of all possible algorithms by $\mathbb{N}$. Then, the set of algorithms that halt on input 0 is semi-decidable but not decidable [Fer09, pp. 5–6]. This problem, called the *Halting Problem*, has proved to be very influential in the theory of computability.
- The set of algorithms that halt on every given input is not decidable [Fer09, p. 160]. Moreover, one can show that it is not even semi-decidable.

```
Input   : x ∈ ℕ^k
Output: 1 if x ∈ X ∩ Y, 0 otherwise
if A_X(x) = 1 and A_Y(x) = 1 then
  | Return 1
else
  | Return 0
end
```

Algorithm 3: Deciding $X \cap Y$

Lemma 2.1.11 (inheritance properties for computable subsets)**.** *Let $X, Y \subset \mathbb{N}^k$ be decidable. Then, also $X \cap Y$, $X \cup Y$ and $\mathbb{N}^k \backslash X$ are decidable.*

Proof. We prove the statement for $X \cap Y$. The other statements then follow similarly. Let A_X be an algorithm which decides X and A_Y be an algorithm deciding Y. Then, the Algorithm 3 decides $X \cap Y$. □

Semi-decidable sets can be listed by an algorithm, which also explains the name *recursively enumerable.*

Lemma 2.1.12 (semi-decidable vs. recursive enumeration, [Cut80, Theorem 7.2.7])**.** *A set $X \subset \mathbb{N}^k$ is semi-decidable if and only if there is an algorithm that enumerates all elements of X in no particular order, i.e. there is an algorithm that computes a function $f : \mathbb{N} \to \mathbb{N}^k$ such that*

$$X = \{f(x) \mid x \in \operatorname{dom} f\}.$$

However, in general, we cannot control the order in which these elements are listed. On the other hand, decidable sets can be listed in an increasing order.

Lemma 2.1.13 (decidable vs. enumeration)**.** *A subset $X \subset \mathbb{N}$ is decidable if and only if it can be listed in increasing order, i.e. if there is a (partial) computable function $f : \mathbb{N} \to \mathbb{N}$ that is (on its domain) monotonically increasing and*

$$X = \{f(x) \mid x \in \operatorname{dom} f\}.$$

Proof (Sketch). Suppose that X is decidable. We have that X as well as $\mathbb{N} \backslash X$ are semi-decidable [Cut80, Theorem 7.2.6]. Thus, the above Lemma 2.1.12 yields that for both sets, there are algorithms that list their elements (in an arbitrary order). From this, we can construct an algorithm listing all elements of X in an increasing order (by waiting until all smaller numbers have been listed by one of the two algorithms).

Conversely, if f is monotonically increasing, and lists X, we can decide X as follows: Without loss of generality, we assume that $\operatorname{dom} f$ is infinite. Otherwise, X is decidable because it is finite (Example 2.1.10). For $n \in \mathbb{N}$, we start computing the images of f in increasing order. If we obtain n, then, we know that $n \in X$. If we reach first a number bigger than n, we can conclude that $n \notin X$. □

2.2 Computability Classes of Real Numbers

In this section, we introduce different notions of computability of real numbers. We will mainly follow the article of Zheng and Rettinger [ZR04].

Definition 2.2.1 (computable sequences)**.** A *computable sequence* is a sequence $(q_n)_{n\in\mathbb{N}}$ that is computable as a (total) function $\mathbb{N} \to \mathbb{Q}$.

Definition 2.2.2 (computability classes)**.** Let $r \in \mathbb{R}$.

- We say that r is *effectively computable* (or just *computable*) if there exists a computable sequence $(q_n)_{n\in\mathbb{N}}$ such that for all $n \in \mathbb{N}$, we have

  $$|r - q_n| \leq 2^{-n}.$$

  We denote the set of effectively computable numbers by EC.

- We say that r is *left (resp. right-)computable* (or *lower (resp. upper) semi-computable*) if there exists a computable sequence $q : \mathbb{N} \to \mathbb{Q}$ that is monotonically increasing (resp. decreasing) such that

  $$r = \lim_{n\to\infty} q_n.$$

  We denote the set of left-(resp. right-) computable numbers by LC (resp. RC).

- We set $\mathrm{SC} := \mathrm{LC} \cup \mathrm{RC}$ the set of *semi-computable* non-negative numbers.

- We say that r is *weakly computable* if it is the difference of two left-computable real numbers. We denote the set of these numbers by WC.

- We say that r is *computably approximable* (or *computably approachable* or *limit computable*) if there is a computable sequence $(q_n)_{n\in\mathbb{N}}$ of rational numbers such that

  $$\lim_{n\to\infty} q_n = r$$

  and denote the set of computably approximable numbers by CA.

Moreover, we write $\mathrm{EC}_{\geq 0}, \mathrm{LC}_{\geq 0}$, etc. for the non-negative numbers in these sets.

Remark 2.2.3. The above definition talks about existence of computable sequences. This does *not* necessarily mean that in every case, we can point to a specific sequence that satisfies the desired properties.

Proposition 2.2.4. *We have the following relations between these classes:*

$$\mathrm{EC} = \mathrm{LC} \cap \mathrm{RC} \subsetneq \mathrm{LC} \cup \mathrm{RC} = \mathrm{SC} \subsetneq \mathrm{WC} \subsetneq \mathrm{CA}.$$

Proof. see [AWZ00] □

Many numbers that we encounter are effectively computable.

Example 2.2.5 (effectively computable numbers)**.**

- Every rational number is effectively computable. This follows from the fact that constant sequences are computable.
- Moreover, even every algebraic number is effectively computable. One can see this as follows: Let $r \in \mathbb{R}$ be an algebraic number and let $p \in \mathbb{Q}[x]$ be the minimal polynomial of r. Because algebraic extensions over $\mathbb{Q}$ are separable, p has a single root at r, i.e. there exist $q \in \mathbb{Q}$ and $n_0 \in \mathbb{N}$, such that $q < r < q + 2^{-n_0}$, the polynomial p has no other root in $[q, q + 2^{-n_0}]$ and $p(q)$ and $p(q + 2^{-n_0})$ have different signs. We can then recursively divide $[q, q + 2^{-n_0}]$ into two halves. The algorithm then determines the half where we have the sign change (which can be done by evaluating at both end points). We iterate and divide this interval again, and we always consider the half that contains the change of signs. Then, the sequence of the middle points effectively converges to the root of p in $[q, q + 2^{-n_0}]$, hence r.
- Moreover, the transcendental numbers π and e are also effectively computable.

  For Euler's number e, one can see this as follows: We have $e = \sum_{k=0}^{\infty} \frac{1}{k!}$. For the approximating sequence $S_n := \sum_{k=0}^{n} \frac{1}{k!}$, we have

  $$|e - S_n| \leq \frac{1}{n!}.$$

  Then, $e \in \mathrm{EC}$ follows from Proposition 2.2.7 below.

  For π, one can use a similar argument, for instance using the *Gregory-Leibniz series* $\pi = 4 \cdot \sum_{k=0}^{\infty} \frac{(-1)^k}{2k+1}$.
- The set EC is a field (with respect to the usual addition and multiplication in $\mathbb{R}$) [Ric54, Theorem 4]. The same statement is *not* true for LC or RC. (See Proposition 2.2.9 and the fact that complements of semi-decidable sets are not necessarily semi-decidable (Example 2.1.10))
- Let $A \subset \mathbb{N}$ be a semi-decidable but *not* decidable set. For instance, A could be a set obtained from the Halting Problem (see Example 2.1.10). Then, we have

  $$\sum_{n \in A} 2^{-n} \in \mathrm{LC} \backslash \mathrm{EC}$$

  by Proposition 2.2.9 and Proposition 2.2.7.

However, even though many examples that we usually come into contact with are contained in EC, it is a rather small class of real numbers from the viewpoint of cardinality.

Lemma 2.2.6. *All the classes of real numbers introduced in the above Definition 2.2.2 are countably infinite.*

Proof. By Proposition 2.2.4, it suffices to show that EC is infinite and that CA is at most countable.

The set EC is infinite because it contains $\mathbb{Q}$ (Example 2.2.5). The fact that CA is countable can be shown as follows: For every element $r \in \text{CA}$ there exists a computable sequence $(q_n)_{n\in\mathbb{N}}$ that converges to r. This sequence is computed by an algorithm, which uniquely determines r. Because the set of algorithms is countable, so is CA. □

Effective computability can be characterised as follows:

Proposition 2.2.7 (effective computability)**.** *Let $r \in \mathbb{R}$. Then, the following are equivalent:*

1. $r \in \text{EC}$.
2. *There are computable sequence $(q_n)_{n\in\mathbb{N}}$ and $(\epsilon_n)_{n\in\mathbb{N}}$ of rational numbers such that $\lim_{n\to\infty} \epsilon_n = 0$ and for all $n \in \mathbb{N}$, we have*
$$|r - q_n| \leq \epsilon_n.$$
3. *The Dedekind cut*
$$L_r := \{x \in \mathbb{Q} \mid x < r\}$$
is a decidable set.
4. *There exist $k \in \mathbb{Z}$ and a decidable subset $A \subset \mathbb{N}_{>0}$ such that*
$$r = k + \sum_{n\in A} 2^{-n}$$

The equivalence of 1, 3 and 4 is stated in [ZR04, Theorem 1.1]. Nevertheless, we will give a sketch of a proof.

Proof (sketch). By definition of effective computability, 1 implies 2, using the computable sequence $\epsilon_n := 2^{-n}$.

To show that 2 implies 3, suppose that condition 2 holds. Without loss of generality, we can assume that $r \notin \mathbb{Q}$ (as rational numbers clearly satisfy 3). Thus, we have to show that we can algorithmically decide whether any $x \in \mathbb{Q}$ satisfies $x < r$ or $x > r$ (the case $x = r$ is not possible because $r \notin \mathbb{Q}$ by assumption). We can then simultaneously list all elements of the sequences $(q_n + \epsilon_n)_{n\in\mathbb{N}}$ and $(q_n - \epsilon_n)_{n\in\mathbb{N}}$. Note that the former converges to r from above while the latter converges r from below.

If we find $n \in \mathbb{N}$ with $q_n + \epsilon_n < x$, then, we have $r < x$, whereas if we find $n \in \mathbb{N}$ with $q_n - \epsilon_n > x$, we have $r > x$. By the convergence, one of these cases will occur, hence we can decide whether $x < r$.

Next, suppose that condition 3 holds. For simplicity, assume $r \in [0, 1)$ and set $k = 0$. Then, there exists a binary expansion of r, i.e. a set $A \subset \mathbb{N}_{>0}$ such that

$$r = \sum_{n \in A} 2^{-n}.$$

We need to show that A is a decidable set. By Lemma 2.1.13, it suffices to show that we can list A in increasing order. If $n \in \mathbb{N}$, and $A \cap \{1, \ldots, n\}$ is already determined, we can check if

$$\left(\sum_{k \in A \cap \{1,\ldots,n\}} 2^{-k} \right) + 2^{-(n+1)} \in L_r,$$

If this is the case, then $n + 1 \in A$, otherwise $n + 1 \notin A$.

Finally, we show that 4 implies 1: If the situation in 4 is satisfied, then

$$q_n := k + \sum_{l \in A \cap \{1,\ldots,n\}} 2^{-l}$$

satisfies the claim in 1 and this sequence is computable by the fact that A is decidable. □

Remark 2.2.8. Note that there is no efficient way to pass between all of these four different descriptions the crucial point is the passage from 2 to 3 where we there is a canonical way to construct an algorithm deciding L_r if $r \notin \mathbb{Q}$. However, we cannot determine algorithmically from the situation in 2 whether $r \in \mathbb{Q}$. Even if we know that $r \in \mathbb{Q}$, we cannot algorithmically determine q and hence we cannot algorithmically construct an algorithm deciding L_r.

Similarly, there is the following characterisation for left- and right-computable numbers, which can be proved analogously.

Proposition 2.2.9 (left and right computability)**.** *Let $r \in \mathbb{R}$. Then, the following are equivalent:*

1. $r \in \mathrm{LC}$ *(resp.* $r \in \mathrm{RC}$*)*
2. *There exists a computable sequence* $(q_n)_{n \in \mathbb{N}}$ *of rational numbers such that we have* $\lim_{n \to \infty} q_n = r$ *and*

$$r = \sup_{n \in \mathbb{N}} q_n \quad \left(\textit{resp. } r = \inf_{n \in \mathbb{N}} q_n \right).$$

3. *The Dedekind cut*

$$L_r := \{x \in \mathbb{Q} \mid x < r\} \quad \Big(\textit{resp. } R_r := \{x \in \mathbb{Q} \mid x > r\}\Big)$$

is a semi-decidable set.

4. *There exist $k \in \mathbb{Z}$ and a semi-decidable subset $A \subset \mathbb{N}_{>0}$ (resp. a subset $A \subset \mathbb{N}_{>0}$ such that $\mathbb{N}_{>0} \setminus A$ is semi-decidable) such that*

$$r = k + \sum_{n \in A} 2^{-n}.$$

2.3 Right-Computability of some Topological Invariants

In this section, we will take a look at two examples where topological invariants are right-computable real numbers.

2.3.1 Stable Commutator Length

We will take a closer look at the following result by Nicolas Heuer.

Theorem 2.3.1 (scl is right-computable, [Heu19, Theorem A])**.** *Denote by* $\mathrm{SCL}^{\mathrm{rp}}$ *the set of stable commutator lengths arising from recursively presented groups. We have*

$$\mathrm{SCL}^{\mathrm{rp}} = \mathrm{RC}_{\geq 0}.$$

We recall first the relevant definitions.

Definition 2.3.2 (recursively presented group)**.** A group G is called *recursively presented* if it admits a presentation

$$G \cong \langle s_0, s_1, s_2, \cdots \mid R \rangle$$

where the set of relations $R \subset F(s_0, s_1, s_2, \dots)$ is semi-decidable. Here, we denote by $F(s_0, s_1, s_2, \dots)$ the set of all reduced words in $s_0, s_1, s_2, \dots$.

One import fact here is that the word problem is semi-decidable. More precisely, the following holds.

Lemma 2.3.3. *Let*

$$G = \langle s_0, s_1, s_2, \cdots \mid R \rangle$$

be a recursive presentation. The words representing the trivial element in G *are exactly those in the normal closure* $\langle\langle R \rangle\rangle$*. Because* R *is recursively enumerable, also its normal closure is recursively enumerable.*

In the finitely generated case, Higman's embedding theorem characterises recursively presented groups.

Theorem 2.3.4 (Higman's Embedding Theorem, [Hig61])**.** *A finitely generated group* G *is recursively presented if and only if there is a finitely presented group* H *such that* $G \subset H$ *is a subgroup.*

Moreover, we recall the notion of stable commutator length.

Definition 2.3.5 (stable commutator length)**.** Let G be a group.

- We define the *commutator group* of G to be the subgroup $[G, G]$ of G generated by the *commutators*, i.e. by the elements
$$[g, h] := ghg^{-1}h^{-1}$$
for all $g, h \in G$.
- For an element $g \in [G, G]$, we define the *commutator length* by
$$\mathrm{cl}_G(g) := \min\{n \in \mathbb{N} \mid \exists_{x_1,\dots,x_n,y_1,\dots,y_n \in G} \;\; g = [x_1, y_1] \cdots [x_n, y_n]\}$$
- We define the *stable commutator length* of $g \in [G, G]$ by
$$\mathrm{scl}_G(g) := \lim_{n\to\infty} \frac{\mathrm{cl}_G(g^n)}{n} \in \mathbb{R}_{\geq 0}.$$

Remark 2.3.6. Note that in the definition of stable commutator length, the convergence of the sequence $\left(\frac{\mathrm{cl}_G(g^n)}{n}\right)_{n\in\mathbb{N}}$ is from above, i.e. the limit is an infimum. This fact will play an essential role in the proof that the stable commutator length is a right-computable real number.

For the spirit of this thesis, the following (easier) inclusion of Heuer's theorem is most important.

Proposition 2.3.7. *Let G be a recursively presented group, $g \in [G, G]$. Then,*
$$\mathrm{scl}_G(g) \in \mathrm{RC}_{\geq 0}.$$

Proof. By definition, we have $\mathrm{scl}_G(g) \geq 0$. By Proposition 2.2.9, we need to show that the set
$$\{x \in \mathbb{Q} \mid x > \mathrm{scl}_G(g)\}$$
is semi-decidable. We follow the argument of Heuer [Heu19, Chapter 6].

Fix a word $w \in F(s_0, s_1, \dots)$ that represents g. Let $x \in \mathbb{Q}$. Without loss of generality, we assume $x > 0$. Because the limit in the definition of scl is an infimum, we have $x > \mathrm{scl}_G(g)$ if and only if there is $N \in \mathbb{N}_{>0}$ such that
$$x > \frac{\mathrm{cl}_G(g^N)}{N},$$
or equivalently $k := \lceil N \cdot x \rceil - 1 \in \mathbb{N}$ satisfies
$$k \geq \mathrm{cl}_G(g^N).$$

But this is the case if and only if there exist $x_1, \dots, x_k, y_1, \dots, y_k \in F(s_0, s_1, \dots)$ and $r \in \langle\langle R \rangle\rangle$ such that

$$w^n = [x_1, y_1] \cdots [x_k, y_k] \cdot r$$

in $F(s_0, s_1, \dots)$. Since $F(s_0, s_1, \dots)$ is countable, $\langle\langle R \rangle\rangle$ is recursively enumerable and we have to check this for all $N \in \mathbb{N}_{>0}$, we can enumerate all possibilities on the right-hand side to semi-decide this problem. □

Remark 2.3.8 (the other inclusion). The other (more difficult) inclusion, i.e. $\mathrm{SCL}^{\mathrm{rp}} \supseteq \mathrm{RC}_{\geq 0}$ is proved by giving an explicit recursive presentation and an element of the group and then using an argument involving van Kampen diagrams on admissible surfaces to determine the stable commutator length. For details, see Heuer's article [Heu19, Chapter 5].

2.3.2 Simplicial Volume

We will take a look at another topological invariant. Consider the following result.

Theorem 2.3.9 (Right-Computability of Simplicial Volume, [HL20, Theorem E]). *Let M be an oriented, closed, connected manifold. Then,* $||M|| \in \mathrm{RC}_{\geq 0}$.

Here, $||M||$ denotes the *simplicial volume*. Recall that it is defined as follows.

Definition 2.3.10 (simplicial volume, [Löh11, Definition 1.1]). Let M be an oriented, closed, connected manifold of dimension $n \in \mathbb{N}$. Recall that $H_n(M, \mathbb{R}) \cong \mathbb{R}$ has a distinguished generator $[M]$, called the *fundamental class*, given by the orientation of M. Then, the *simplicial volume* of M is defined as

$$||M|| := \inf\{|c|_1 \mid c \in C_n(M, \mathbb{R}) \text{ represents } [M]\} \in \mathbb{R}_{\geq 0}.$$

Here, $|c|_1$ denotes the ℓ^1-norm on $C_n(M, \mathbb{R})$ which is defined as follows: If $c = \sum_{i=0}^{k} a_i \cdot \sigma_i$ is in reduced form, we define

$$|c|_1 = \sum_{i=0}^{k} |a_i| \in \mathbb{R}_{\geq 0}.$$

More generally, for $\alpha \in H_n(M, \mathbb{R})$, we define

$$||\alpha|| := \inf\{|c|_1 \mid c \in C_n(M, \mathbb{R}) \text{ represents } \alpha\} \in \mathbb{R}_{\geq 0}.$$

Moreover, we define similar notions with $\mathbb{Z}$-coefficients.

The following two lemmas are key in proving the theorem.

Lemma 2.3.11 (simplicial volume via simplicial complexes, [HL20, Chapter 4.2])**.** *Let M be an oriented, closed, connected manifold. Then, M is homotopy equivalent to a finite simplicial complex T. Moreover, we have*

$$||M|| = \inf_{m \in \mathbb{N}_{>0}} \frac{||m \cdot [T]_{\mathbb{Z}}||_{\mathbb{Z}}}{m}.$$

Here, $[T]_{\mathbb{Z}}$ denotes a generator of the singular homology $H_n(|T|, \mathbb{Z})$.

Lemma 2.3.12 (simplicial volume via combinatorial singular chains, [HL20, Proof of Lemma 4.5])**.** *Let T be a finite simplicial complex and $\alpha \in H_n(|T|, \mathbb{Z})$. Then,*

$$||\alpha||_{\mathbb{Z}} = \inf\{|c|_1 \mid c \in C_n(|T|, \mathbb{Z}) \text{ represents } \alpha \text{ and } c \text{ is a combinatorial singular chain}\}.$$

Here, being a combinatorial singular chain *means that $c = \sum_{i=0}^{k} a_i \cdot \sigma_i$, where $a_i \in \mathbb{Z}$, and, after barycentric subdivisions of Δ^n and T, the σ_i are simplicial maps $\Delta^n \to T$.*

Combining these two lemmas, we obtain the following result.

Lemma 2.3.13. *Let M be an oriented, closed, connected manifold. Let T be a finite, simplicial complex that is homotopy equivalent to M. Let C be the set of all closed combinatorial singular n-chains of T. Moreover, for $c \in C$, let $m(c) \in \mathbb{Z}$ be the unique integer such that $[c] = m(c) \cdot [T]_{\mathbb{Z}}$ in $H_n(|T|, \mathbb{Z})$. Then, we have*

$$||M|| = \inf_{c \in C, m(c) > 0} \frac{|c|_1}{m(c)}.$$

We can now prove the theorem, following the ideas of Heuer and Löh [HL20, Chapter 4.2].

Proof of Theorem 2.3.9. By the above Lemma 2.3.13, we can write $||M||$ as an infimum of a countable set of rational numbers. By Proposition 2.2.9, in order to show right-computability, it suffices to show that the set C of all closed, combinatorial singular n-chains of T is recursively enumerable and that for a given $c \in C$, we can compute $|c|_1$ as well as $m(c)$.

We can enumerate all closed, combinatorial singular n-chains of T as we can iteratively enumerate all subdivisions of Δ^n and T. Moreover, we can enumerate all simplicial maps between two finite, simplicial complexes. Furthermore, we can algorithmically check whether such a chain is closed.

Given $c \in C$, we can compute $|c|_1$ directly (if we allow for chains occurring in different forms, we might not even need to pass to the reduced form). Moreover, for a given $m \in \mathbb{N}_{>0}$, we can check whether c represents $m \cdot [T]_{\mathbb{Z}}$ by comparison with the iterated barycentric subdivision in simplicial homology.

Thus, we obtain that $||M|| \in \mathrm{RC}_{\geq 0}$. □

2.3.3 Comparison with the Case of ℓ^2-Betti Numbers

In the main part of this thesis (Chapter 3), we will prove that ℓ^2-Betti numbers are (under certain assumptions) (right-)computable.

In the previous two examples, we have seen that it is a good strategy to write the number in question as an infimum of a computable sequence. For ℓ^2-Betti numbers, the first statement that seems suitable for this is Lück's approximation theorem (Theorem 1.3.9). Recall that it states that

$$b_n^{(2)}(G \curvearrowright X) = \lim_{i\to\infty} \frac{b_n(G_i\backslash X)}{[G : G_i]}$$

for a free, finite type G-CW complex X and a residual chain $(G_i)_{i\in\mathbb{N}}$ of G. However, this limit is not always the infimum, as the following example shows.

Example 2.3.14. Consider the finite type CW complex

$$X := \mathbb{R}P^\infty \vee (\mathbb{Z}/3\mathbb{Z})\backslash S^\infty$$

where $S^\infty = \operatorname{colim}_{n\in\mathbb{N}} S^n$, and $\mathbb{R}P^\infty$ is the quotient by the antipodal action of $\mathbb{Z}/2\mathbb{Z}$ on S^∞, and $\mathbb{Z}/3\mathbb{Z} \curvearrowright S^\infty$ is a free, cellular action.

We have $\pi_1(X) \cong \mathbb{Z}/2\mathbb{Z} * \mathbb{Z}/3\mathbb{Z} \cong \mathrm{PSL}(2,\mathbb{Z})$, which is a residually finite group. Moreover, we have [Kam19, p. 78]

$$b_1^{(2)}(\pi_1(X) \curvearrowright \widetilde{X}) = \frac{1}{6}$$

but for the first element $G_0 := \pi_1(X)$ in a residual chain of $\pi_1(X)$, we have

$$\frac{b_1(X)}{[\pi_1(X) : G_0]} = \frac{0}{1} = 0 < \frac{1}{6}.$$

In particular, the convergence in Lück's approximation theorem is *not* from above.

Hence, in the following chapter, we will need a different sequence that converges from above to the ℓ^2-Betti number. Such a sequence will be exhibited in Chapter 3.2. On the other hand, we can stick with the sequence from the Lück approximation theorem: In this case, we need to bound the error between the approximation term and the ℓ^2-Betti number. If this bound is computable and tends to zero, we can add it to the approximation term on the right hand side of Lück's approximation theorem to obtain a sequence that converges from above to the desired ℓ^2-Betti number. We will develop this approach (and even prove effective computability) in Chapter 3.4.

3 Computability of ℓ^2-Betti numbers

3.1 Some Known Results

In this chapter, we will present some results that are already known in the literature about computability of ℓ^2-Betti numbers.

For instance, Grabowski proved the following theorem.

Theorem 3.1.1 ([Gra14, Theorem 1.4]). *Every non-negative, effectively computable real number occurs as an ℓ^2-Betti number of a finitely presented group, i.e. we have* $\mathrm{EC}_{\geq 0} \subseteq \widetilde{B}^{(2)}$.

Later, this result was independently proved by Pichot, Schick and Zuk in the following form. It is slightly stronger, which can be seen using Higman's embedding theorem (see Theorem 2.3.4).

Theorem 3.1.2 ([PSZ15, Theorem 11.1, Remark 13.3]). *Every non-negative, effectively computable real number occurs as an ℓ^2-Betti number of a finitely generated, recursively presented group with solvable word problem.*

Remark 3.1.3. The proof given by Pichot, Schick and Zuk is quite explicit, i.e. given a decidable set $I \subset \mathbb{N}$, there is an explicit construction of a group G_I and an element $A \in \mathbb{Z}G_I$ such that

$$\dim_{\mathcal{R}G_I} \ker(\cdot A) = \sum_{n \in I} 2^{-n}.$$

Remark 3.1.4. The key idea of the work by Pichot, Schick and Zuk was to construct an operator that 'accepts' local patterns in the Cayley graph of Γ, where Γ is either the free group of rank two or $\Gamma = \mathbb{Z} \wr \mathbb{Z}$. In the present case, so called 'hooks' were considered. This allows to decompose the operator in question nicely, so that it is possible to calculate the dimension of an eigenspace explicitly.

Moreover, if we drop the condition on the solvability of the word problem, we obtain the following result:

Theorem 3.1.5. *Every non-negative, weakly computable real number occurs as an ℓ^2-Betti number of a finitely presented group, i.e.* $\mathrm{WC}_{\geq 0} \subseteq \widetilde{B}^{(2)}$.

Proof. It suffices to show the claim for finitely generated, recursively generated groups since by Higman's embedding theorem (Theorem 2.3.4), every such group embeds into a finitely presented group and moreover, ℓ^2-Betti numbers are monotone under inclusions (Proposition 1.2.23).

Since sums of ℓ^2-Betti numbers are ℓ^2-Betti numbers (Proposition 1.2.23) and products of two finitely generated, recursively presented groups have again these properties, it suffices to show that all right-computable and all left-computable nonnegative numbers occur. For right-computable numbers, this is stated in [PSZ15, Remark 13.5]. For left-computable numbers, there are $\beta_1'', \beta_2'' \in \mathbb{Q}, d \in \mathbb{N}$ such that for every recursively enumerable set $I \subset \mathbb{N}$, we have that

$$\beta_1'' - \beta_2'' \cdot \sum_{k \in I} 2^{-dk}$$

is an ℓ^2-Betti number of a finitely generated, recursively presented group [PSZ15, Remark 13.6]. Then, by the fact that we can take sums and scale by rational numbers (Proposition 1.2.23), we obtain all nonnegative left-computable numbers from this. □

Conversely, there are also results on computability restrictions of ℓ^2-Betti numbers, for instance the following one that was first proved in the Bachelor's thesis of T. Groth [Gro12, Theorem 6.12].

Theorem 3.3.23. *Let b be an ℓ^2-Betti numbers arising from a finitely presented group with solvable word problem that satisfies the determinant class conjecture (Conjecture 3.3.21). Then, b is effectively computable.*

We will give a proof of this statement in Chapter 3.3. Note that the determinant class conjecture is satisfied for a large class of groups, for instance by sofic groups (see Definition 3.3.3), hence in particular by amenable or residually finite groups. This theorem, together with the Theorem 3.1.1 at the beginning of this chapter, imply the following corollary.

Corollary 3.1.6. *The ℓ^2-Betti numbers arising from all groups with solvable word problem that satisfy the determinant conjecture are exactly the effectively computable, non-negative real numbers.*

In addition, Pichot, Schick and Zuk mention the following result by Groth:

Theorem 3.1.7 ([Gro12, Chapter 6.4] as cited in [PSZ15, Remark 13.4])**.** *Let $r \in \widetilde{B}^{(2)}$. Then, there exists a computable sequence $(q_n)_{n \in \mathbb{N}}$ such that*

$$r = \limsup_{n \to \infty} q_n.$$

Together with Theorem 3.1.5, we know that the set of ℓ^2-Betti numbers arising from finitely presented groups contains $\mathrm{WC}_{\geq 0}$ and is contained in the set of real numbers occurring as the limit superior of computable sequences. It is thus natural to ask the following open question.

Question 3.1.8. What does the set $\widetilde{B}^{(2)}$ of ℓ^2-Betti numbers arising from finitely presented groups exactly look like?

Note that this set is countable, because there exist only countable many isomorphism types of finitely presented groups and these groups contain at most countably many elements.

We will finish this chapter by presenting a surprising result about the undecidability of trivial kernels.

Theorem 3.1.9 ([Gra15b, Corollary 6.2])**.** *Consider the group*

$$G := \langle a, t, s \mid a^2 = 1, [t, s] = 1, [t^{-1}at, a] = 1, s^{-1}as = at^{-1}at \rangle.$$

This group has a decidable word problem. Moreover, the problem Zero-divisors-in-$\mathbb{Z}G$ *is undecidable, i.e. the following problem is undecidable:*

- *Input: An element* $T \in \mathbb{Z}G$
- *Output: 'Yes' if* $\ker T \neq 0$*, 'No' otherwise.*

Remark 3.1.10. It suffices to prove the undecidability of the zero divisors problem for $(\mathbb{Z}/2\mathbb{Z}) \wr \mathbb{Z}$ instead, since this group is a subgroup of the group G. The key idea of Grabowski was to embed Turing machines into the group ring. That way, one can reduce the problem whether for a given Turing machine there exists an input which is accepted (which is known to be undecidable) to the zero divisors problem, thus proving that also this problem is undecidable.

Remark 3.1.11 (a contradiction to effective computability?!). At first glance, the above Theorem 3.1.9 seems to contradict Corollary 3.3.13. Since the group G in the above Theorem 3.1.9 is finitely presented, sofic and has solvable word problem, there is an algorithm that, for every $A \in \mathbb{Z}G$, (effectively) computes $\dim_{\mathcal{R}G} \ker(\cdot A)$. By faithfulness, $\ker(\cdot A) = 0$ if and only if its dimension is zero.

Now, the naive approach to solving the zero-divisor problem in $\mathbb{Z}G$ would be the following: Given $A \in \mathbb{Z}G$, (effectively) compute $\dim_{\mathcal{R}G} \ker(\cdot A)$. If it is zero, output 'no', otherwise 'yes'.

If this worked, it would then contradict Theorem 3.1.9. However, there is an error in the argument: This is due to the fact that effective computability does not mean that we 'know' the precise value but rather that we make the deviation arbitrarily small. This confusion arises especially if the notion of effective computability is instead just called 'computability'. This is also one reason why in this thesis, we use the term 'effective computability'.

Remark 3.1.12 (decidable zero-divisors problem). In the presence of additional assumptions that guarantee the existence of an $\epsilon > 0$ such that $\dim_{\mathcal{R}G} \ker(\cdot A)$ cannot take values in $(0, \epsilon)$, the zero-divisor-in-$\mathbb{Z}G$ problem is decidable.

For instance, such assumptions are satisfied by finitely generated sofic groups that have a uniform bound on the order of finite subgroups (so-called BFS groups), and that satisfy the Atiyah conjecture [Gra15b, Proposition 3.1].

3.2 The General Case

In this section, we establish computability results for ℓ^2-Betti numbers of finitely generated groups with solvable word problem.

3.2.1 Right-Computability

We start with a result on right-computability.

Theorem 3.2.1 (Right-computability)**.** *Let G be a finitely generated group with solvable word problem. Moreover, let $A \in M_{n\times n}(\mathbb{Z}G)$ be self-adjoint. Then,*

$$\dim_{\mathcal{R}G} \ker(\cdot A) \in \mathrm{RC}_{\geq 0}.$$

For completeness, we recall the *solvability of the word problem.*

Definition 3.2.2 (solvable word problem)**.** Let G be a finitely generated group with finite generating set $S \subset G$. We say that G *has a solvable word problem* if the set

$$\{w \in F(S) \mid w = e \text{ in } G\}$$

is decidable, i.e. if there exists an algorithm that, given a word $w \in F(S)$, decides whether w represents the trivial element in G.

Remark 3.2.3. The solvability of the word problem is independent of the fixed generating set, i.e. if it is solvable with respect to one finite generating set, it is so with respect to any finite generating set.

In order to prove the theorem above, we fix the following setup:

Setup 3.2.4. Let G be a finitely generated group with solvable word problem. Moreover, let $A \in M_{n\times n}(\mathbb{Z}G)$ be self-adjoint. We denote by μ_A the spectral measure of A (see Definition 1.3.15). We view this measure supported on an interval $[0, d]$ with $d := ||\cdot A|| \geq 0$. We assume without loss of generality that $d \geq 1$ (since we can always make d bigger).

Remark 3.2.5. Note that $\mu_A\big([0, d]\big) = \mathrm{tr}_{\mathcal{R}G}(A^0) = \mathrm{tr}_{\mathcal{R}G}(\mathrm{id}_n) = n$ by Proposition 1.3.16.

Lemma 3.2.6. *In the situation of Setup 3.2.4, we have for all $k \in \mathbb{N}$ that*

$$\dim_{\mathcal{R}G} \ker(\cdot A) \leq \mathrm{tr}_{\mathcal{R}G}\left(\left(1 - \frac{1}{d}\cdot A\right)^k\right).$$

Proof. We have

$$\begin{aligned}
\dim_{\mathcal{R}G} \ker(\cdot A) &= \mu_A(\{0\}) && \text{(Proposition 1.3.17)}\\
&\leq \int_0^d \Big(1 - \frac{1}{d}\cdot x\Big)^k \, d\mu_A && \text{(monotonicity of measures)}\\
&= \operatorname{tr}_{\mathcal{R}G}\bigg(\Big(1-\frac{1}{d}\cdot A\Big)^k\bigg) && \text{(Proposition 1.3.16)},
\end{aligned}$$

as desired. □

Moreover, we have the following lemma about convergence.

Lemma 3.2.7. *In the situation of Setup 3.2.4, we have*

$$\dim_{\mathcal{R}G} \ker(\cdot A) = \lim_{k\to\infty} \operatorname{tr}_{\mathcal{R}G}\bigg(\Big(1-\frac{1}{d}\cdot A\Big)^k\bigg).$$

Proof. We have

$$\begin{aligned}
\lim_{k\to\infty} \operatorname{tr}_{\mathcal{R}G}\bigg(\Big(1-\frac{1}{d}\cdot A\Big)^k\bigg) &= \lim_{k\to\infty} \int_0^d \Big(1 - \frac{1}{d}\cdot x\Big)^k \, d\mu_A && \text{(Proposition 1.3.16)}\\
&= \int_0^d \lim_{k\to\infty} \Big(1 - \frac{1}{d}\cdot x\Big)^k \, d\mu_A && \text{(dominated convergence)}\\
&= \int_0^d \chi_{\{0\}} \, d\mu_A \\
&= \mu_A(\{0\}) \\
&= \dim_{\mathcal{R}G} \ker(\cdot A) && \text{(Proposition 1.3.17)},
\end{aligned}$$

where we used the theorem of dominated convergence. In order to apply this theorem, note that for all $k \in \mathbb{N}$ and $x \in [0, d]$, we have

$$\left|\Big(1 - \frac{1}{d}\cdot x\Big)^k\right| \leq \chi_{[0,d]}(x),$$

hence, the integrable function $\chi_{[0,d]}$ dominates the sequence of functions. □

We can now prove the theorem at the beginning of this section.

Proof of Theorem 3.2.1. We assume the situation of Setup 3.2.4. From Lemma 3.2.6 and Lemma 3.2.7, we obtain that the sequence

$$\bigg(\operatorname{tr}_{\mathcal{R}G}\bigg(\Big(1-\frac{1}{d}\cdot A\Big)^k\bigg)\bigg)_{k\in\mathbb{N}}$$

of rational numbers converges from above to $\dim_{\mathcal{R}G} \ker(\cdot A)$. Moreover, this sequence is computable because for all $k \in \mathbb{N}$,

- We can calculate the matrix $\left(1 - \frac{1}{d} \cdot A\right)^k$.
- We can then calculate the trace of this matrix. (This requires solving the word problem in G.)

Hence, we have $\dim_{\mathcal{R}G} \ker(\cdot A) \in \mathrm{RC}_{\geq 0}$. □

Remark 3.2.8. The obvious question to ask about Theorem 3.2.1 is what happens if we drop the assumption on solvability of the word problem. Theorem 1.3.3 answers this question: Without this assumption, all nonnegative real numbers occur. Moreover, even if we restrict our attention to finitely presented groups that do not necessarily have a solvable word problem, we cannot generalise Theorem 3.2.1 since then, all nonnegative weakly computable numbers occur as ℓ^2-Betti numbers (Theorem 3.1.5) and we have $\mathrm{RC}_{\geq 0} \subsetneq \mathrm{WC}_{\geq 0}$ (Proposition 2.2.4).

3.2.2 Left-Computability

In this section, we apply the same techniques to obtain left-computability. However, in this case, we need an additional assumption. We will see in Section 3.3 that for a large class of groups, this assumption is satisfied.

Theorem 3.2.9 (left-computability)**.** *Let G be a finitely generated group with solvable word problem. Moreover, let $A \in M_{n\times n}(\mathbb{Z}G)$ be self-adjoint. The following are equivalent:*

1. *We have* $\dim_{\mathcal{R}G} \ker(\cdot A) \in \mathrm{LC}_{\geq 0}$.
2. *There exists a computable sequence $(\epsilon_k)_{k\in\mathbb{N}}$ of rational numbers such that $\lim_{k\to\infty} \epsilon_k = 0$ and for all $k \in \mathbb{N}_{>0}$, we have*

$$\forall_{k\in\mathbb{N}} \ \mu_A\Big(\Big(0, \frac{1}{k}\Big)\Big) \leq \epsilon_k.$$

For the proof, we need the following lemma.

Lemma 3.2.10. *Assume the situation of Setup 3.2.4 and let $(\epsilon_k)_{k\in\mathbb{N}}$ be a sequence of rational numbers such that*

$$\forall_{k\in\mathbb{N}} \ \mu_A\Big(\Big(0, \frac{1}{k}\Big)\Big) \leq \epsilon_k.$$

Then, for all $k \in \mathbb{N}$, we have

$$\dim_{\mathcal{R}G} \ker(\cdot A) \geq \mathrm{tr}_{\mathcal{R}G}\left(\left(1 - \frac{1}{d} \cdot A\right)^{k^2}\right) - \epsilon_k - n \cdot \left(1 - \frac{1}{k \cdot d}\right)^{k^2}.$$

Proof. Let $k \in \mathbb{N}$. Then, we have

$$
\begin{aligned}
&\operatorname{tr}_{\mathcal{R}G}\left(\left(1-\frac{1}{d}\cdot A\right)^{k^2}\right) \\
&= \int_0^d \left(1-\frac{x}{d}\right)^{k^2} d\mu_A && \text{(Proposition 1.3.16)} \\
&\leq \mu_A(\{0\}) + \mu_A\big((0,1/k)\big) + \int_{1/k}^d \left(1-\frac{x}{d}\right)^{k^2} d\mu_A \\
&\leq \mu_A(\{0\}) + \mu_A\big((0,1/k)\big) + \mu_A\big([1/k,d]\big)\cdot\left(1-\frac{1/k}{d}\right)^{k^2} && \text{(monotonicity)} \\
&\leq \mu_A(\{0\}) + \epsilon_k + \mu_A\big([0,d]\big)\cdot\left(1-\frac{1}{kd}\right)^{k^2} && \text{(assumption on } \epsilon_k\text{)} \\
&= \dim_{\mathcal{R}G}\ker(\cdot A) + \epsilon_k + \mu_A\big([0,d]\big)\cdot\left(1-\frac{1}{kd}\right)^{k^2} && \text{(Proposition 1.3.17)} \\
&= \dim_{\mathcal{R}G}\ker(\cdot A) + \epsilon_k + n\cdot\left(1-\frac{1}{kd}\right)^{k^2} && \text{(Remark 3.2.5)}
\end{aligned}
$$

where we used that the function $x \mapsto \left(1-\frac{x}{d}\right)^{k^2}$ is bounded by 1 and monotonically decreasing on $[0,d]$. From this inequality, we obtain the claim of the lemma. □

We can now prove the characterisation of left-computability.

Proof of Theorem 3.2.9. Assume first that there is a computable sequence $(\epsilon_k)_{k\in\mathbb{N}}$ as in 2. We denote

$$a_k := \operatorname{tr}_{\mathcal{R}G}\left(\left(1-\frac{1}{d}\cdot A\right)^{k^2}\right) - \epsilon_k - n\cdot\left(1-\frac{1}{k\cdot d}\right)^{k^2}$$

the right hand side in the claim of Lemma 3.2.10. Note that $\lim_{k\to\infty}\epsilon_k = 0$ by assumption and

$$\lim_{k\to\infty}\left(1-\frac{1}{k\cdot d}\right)^{k^2} = 0$$

by elementary calculus. By Lemma 3.2.7 and the fact that taking subsequences preserves the limit, we obtain that

$$\lim_{k\to\infty}\operatorname{tr}_{\mathcal{R}G}\left(\left(1-\frac{1}{d}\cdot A\right)^{k^2}\right) = \dim_{\mathcal{R}G}\ker(\cdot A),$$

Hence, we obtain that

$$\lim_{k\to\infty} a_k = \dim_{\mathcal{R}G}\ker(\cdot A).$$

This convergence is from below by Lemma 3.2.10. Moreover, the sequence $(a_k)_{k\in\mathbb{N}}$ is computable: For this, it suffices to check that all three summands of this sequence are computable (Lemma 2.1.6):

- For $\operatorname{tr}_{\mathcal{R}G}\big((1-\frac{1}{d}\cdot A)^{k^2}\big)$, this follows as in the proof of right-computability (Theorem 3.2.1).
- The sequence $(\epsilon_k)_{k\in\mathbb{N}}$ is computable by assumption.
- The third summand is computable by basic results about computable sequences (Lemma 2.1.6).

Hence, we obtain that $\dim_{\mathcal{R}G}\ker(\cdot A)\in \mathrm{LC}_{\geq 0}$.

Conversely, assume that $\dim_{\mathcal{R}G}\ker(\cdot A)\in \mathrm{LC}_{\geq 0}$, i.e. there exists a computable, monotonically increasing sequence $(a_k)_{k\in\mathbb{N}}$ such that $\lim_{k\to\infty} a_k = \dim_{\mathcal{R}G}\ker(\cdot A)$. We construct a sequence $(\epsilon_k)_{k\in\mathbb{N}}$ as follows: For $k\in\mathbb{N}_{>0}$, we set

$$m(k) := \max\Big\{m\in\mathbb{N} \mid \Big(1-\frac{1}{kd}\Big)^m \geq \frac{1}{2}\Big\}.$$

and

$$\epsilon_k := 2\cdot\Big(\operatorname{tr}_{\mathcal{R}G}\Big(\Big(1-\frac{A}{d}\Big)^{m(k)}\Big) - a_k\Big)$$

Then, the sequence $(\epsilon_k)_{k\in\mathbb{N}}$ is computable, because

- The sequence $\big(m(k)\big)_{k\in\mathbb{N}}$ is computable because for a given $k\in\mathbb{N}_{>0}$, the sequence $\big((1-\frac{1}{kd})^m\big)_{m\in\mathbb{N}}$ is monotonically decreasing and converges to zero for $m\to\infty$. Hence, we can compute $m(k)$ by successively computing all $\big((1-\frac{1}{kd})^m\big)_{m\in\mathbb{N}}$ until we find one element in this sequence that is smaller than $\frac{1}{2}$.
- The trace of matrices in $\mathbb{Z}G$ is computable because G has a solvable word problem.
- The sequence $(a_k)_{k\in\mathbb{N}}$ is computable by assumption.

Moreover, the sequence $(\epsilon_k)_{k\in\mathbb{N}}$ tends to zero for $k\to\infty$. This is because for the sequence $(m(k))_{k\in\mathbb{N}}$, we have $\lim_{k\to\infty} m(k)=\infty$ and thus (see Lemma 3.2.7)

$$\lim_{k\to\infty}\operatorname{tr}_{\mathcal{R}G}\Big(1-\frac{A}{d}\Big)^{m(k)} = \dim_{\mathcal{R}G}\ker(\cdot A) = \lim_{k\to\infty} a_k.$$

Finally, we have to show that for all $k\in\mathbb{N}$ that $\mu_A((0,1/k))\leq\epsilon_k$. This follows from the following computation:

$$\begin{aligned}
\operatorname{tr}_{\mathcal{R}G}\Big(1-\frac{A}{d}\Big)^{m(k)} &= \int_0^d \Big(1-\frac{x}{d}\Big)^{m(k)}\, d\mu_A(x) && \text{(Proposition 1.3.16)}\\
&\geq \int_{\{0\}} \Big(1-\frac{x}{d}\Big)^{m(k)}\, d\mu_A(x)\\
&\quad + \int_{(0,1/k)} \Big(1-\frac{x}{d}\Big)^{m(k)}\, d\mu_A(x)\\
&\geq \mu_A(\{0\}) + \frac{1}{2}\cdot\mu_A((0,1/k)) && \text{(choice of } m(k))
\end{aligned}$$

$$
\begin{aligned}
&= \dim_{\mathcal{R}G} \ker(\cdot A) + \frac{1}{2} \cdot \mu_A((0, 1/k)) && \text{(Proposition 1.3.17)} \\
&\geq a_k + \frac{1}{2} \cdot \mu_A((0, 1/k)) && (a_k \text{ approx. from below})
\end{aligned}
$$

Hence, condition 2 is satisfied. ☐

3.2.3 Effective Computability

Finally, by combining the theorems about right and left computability, we obtain the following corollary.

Corollary 3.2.11 (effective computability)**.** *Let G be a finitely generated group with solvable word problem. Moreover, let $A \in M_{n\times n}(\mathbb{Z}G)$ be self-adjoint. Then, the following are equivalent:*

1. *We have* $\dim_{\mathcal{R}G} \ker(\cdot A) \in \mathrm{LC}_{\geq 0}$.
2. *We have* $\dim_{\mathcal{R}G} \ker(\cdot A) \in \mathrm{EC}_{\geq 0}$.
3. *There exists a computable sequence $(\epsilon_k)_{k\in\mathbb{N}}$ of rational numbers such that* $\lim_{k\to\infty} \epsilon_k = 0$ *and for all $k \in \mathbb{N}_{>0}$, we have*

$$\mu_A\Big(\Big(0, \frac{1}{k}\Big)\Big) \leq \epsilon_k,$$

where we denote by μ_A the spectral measure of A (see Definition 1.3.15).

Proof. This follows by combining Theorem 3.2.9 with Theorem 3.2.1 using the fact that $\mathrm{EC}_{\geq 0} = \mathrm{RC}_{\geq 0} \cap \mathrm{LC}_{\geq 0}$ (Proposition 2.2.4). ☐

Remark 3.2.12. The proof of Theorem 3.2.9 shows that in condition 2, it is equivalently possible to demand that the inequality holds for all $k \geq N_0$ for some $N_0 \in \mathbb{N}$.

3.3 Sofic Groups and the Determinant Class Conjecture

In this chapter, we will deal with ℓ^2-Betti numbers of sofic groups and of groups satisfying the determinant class conjecture.

3.3.1 Sofic groups

The goal of the section is to prove the following theorem.

Theorem 3.3.1. *Let G be a finitely generated, sofic group with solvable word problem, $n \in \mathbb{N}$ and $A \in M_{n\times n}(\mathbb{Z}G)$ be self-adjoint. Then,*

$$\dim_{\mathcal{R}G} \ker(\cdot A) \in \mathrm{EC}_{\geq 0}.$$

Remark 3.3.2. This theorem even holds without the assumption on finite generation, as there is a finitely generated subgroup $H \subseteq G$ such that $A \in \mathbb{Z}H$ (for instance, we can take the subgroup generated by all group elements with non-trivial coefficients in A) and ℓ^2-Betti numbers behave well under inclusions (Proposition 1.2.23).

We will combine the approach of Section 3.2 with the results of Elek and Szabó [ES05]. We then use an argument similar to one given by Grabowski [Gra15b, Claims A2, A3] to conclude.

Proof of Theorem 3.3.1. We will use the characterisation by Corollary 3.2.11. We will see below (Lemma 3.3.12) that the statement 2 of Corollary 3.2.11 is satisfied in this case, yielding that $\dim_{\mathcal{R}G} \ker(\cdot A)$ is effectively computable. □

Before jumping into the details, we review the definition of soficity.

Definition 3.3.3 (sofic group, [ES05])**.** Let G be a finitely generated group, and $S \subseteq G$ be a finite symmetric generating set of G. The group G is *sofic* if there exists a sequence $(V_n, E_n)_{n\in\mathbb{N}}$ of finite directed graphs that are edge-labelled by S and subsets $V_n^0 \subseteq V_n$ such that the following holds:

For all $\delta > 0$ and $r \in \mathbb{N}$, there is $n_{r,\delta} \in \mathbb{N}$ such that for all $m \geq n_{r,\delta}$, we have

- For each $v \in V_m^0$, there is a map

  $$\varphi_v : B_{(G,S)}(r) \longrightarrow B_{(V_m,E_m)}(v, r)$$

  that is an isomorphism of edge-labelled graphs. Here, $B_{(G,S)}(r)$ denotes the r-ball around $e \in \Gamma$ in the Cayley graph of G with respect to the generating set S, labelled by S. Moreover, $B_{(V_m,E_m)}(v, r)$ denotes the r-ball in (V_m, E_m) around v.

- We have $|V_m^0| \geq (1-\delta) \cdot |V_m|$

Remark 3.3.4. The property of being sofic is independent of the chosen finite, symmetric generating set [ES04, Proposition 4.4].

Example 3.3.5 ([ES06, Theorem 2])**.** The class of sofic groups is very large. All residually amenable groups fall into this class, hence in particular amenable groups and residually finite groups are sofic.

It is an open question whether there exist non-sofic groups.

3.3.2 The Spectral Measure near Zero

In this section, we will prove the missing step in the proof of Theorem 3.3.1. More precisely, will will show that there exists a computable bound on $\mu_A((0, 1/k))$, where μ_A is the spectral measure of A.

We thus fix the following setup, following the approach of Elek and Szabó [ES05].

Setup 3.3.6. Let G be a finitely generated, sofic group. Let $S \subset G$ be a finite symmetric generating set of G. Fix $n \in \mathbb{N}$ and $A \in M_{n \times n}(\mathbb{Z}G)$ be self-adjoint. We fix a sequence $(V_n, E_n, V_n^0)_{n\in\mathbb{N}}$ as in Definition 3.3.3.

We define the *operator kernel* as the unique function $K_A : G \times G \to M_{n\times n}(\mathbb{Z})$ such that for all $f \in (\ell^2 G)^n$ and $x \in G$, we have

$$Af(x) = \sum_{y\in G} K_A(x, y) \cdot f(y).$$

More explicitly, for all $x, y \in G$, $K_A(x, y)$ is the $n \times n$-matrix over $\mathbb{Z}$ of all coefficients in A belonging to the group element xy^{-1}.

Let w_A be the *width* of A, i.e. the maximum length of words in S occurring in A. For $m > w_A$ construct the *approximation kernel* $K_A^m : V_m \times V_m \to M_{n\times n}(\mathbb{Z})$ by

$$K_A^m(x, y) := K_A(\gamma, e_G)$$

if $y \in V_m^0$ and $x = \varphi_y(\gamma)$ and $K_A^m(x, y) = 0$ otherwise. Moreover, denote by A_m the bounded linear operator on $[\ell^2(V_m)]^n$ defined by the kernel function K_A^m.

As usual, denote by μ_A the spectral measure of A (Definition 1.3.15). By $(\mu_m)_{m\in\mathbb{N}}$ denote the spectral measures of A_m, scaled by the factor $\frac{1}{|V_m|}$.

Remark 3.3.7. We view μ_A and all $(\mu_m)_{m\in\mathbb{N}}$ as measures on an interval $[0, d]$ for some $d \geq 1$. This is possible because there exists a common bound on the operator norms of A and all $(A_m)_{m\in\mathbb{N}}$ [ES05, Lemma 6.2].

Moreover, all these measures have the same total mass: This follows from the fact that for all $m \in \mathbb{N}$, we have

$$\begin{aligned}
\mu_A([0,d]) &= \operatorname{tr}_{\mathcal{R}G}(I_n) && \text{(Proposition 1.3.16)} \\
&= n \\
&= \frac{n \cdot |V_m|}{|V_m|} \\
&= \frac{\operatorname{tr}(I_{n \cdot |V_m|})}{|V_m|} \\
&= \mu_m([0,d]).
\end{aligned}$$

Elek and Szabó state an approximation theorem [ES05, Proposition 6.1(a)], for which the key lemma is the following:

Lemma 3.3.8 ([ES05, Lemma 6.3])**.** *Assume the situation of Setup 3.3.6. Then, for every polynomial $p \in \mathbb{R}[x]$, we have*

$$\int_0^d p(x)\, d\mu_A(x) = \operatorname{tr}_{\mathcal{R}G}(p(A)) = \lim_{m\to\infty} \frac{\operatorname{tr}(p(A_m))}{|V_m|} = \lim_{m\to\infty} \int_0^d p(x)\, d\mu_m.$$

This implies the following:

Lemma 3.3.9. *The measures $(\mu_m)_{m\in\mathbb{N}}$ converge weakly to μ_A.*

Proof. We have to show that for every continuous function $f : [0,d] \to \mathbb{R}$,

$$\int_0^d f\, d\mu_A = \lim_{m\to\infty} \int_0^d f\, d\mu_m.$$

If f is a polynomial function, this is done in Lemma 3.3.8. For general continuous functions, this can be obtained by uniform approximation with Stone-Weierstraß polynomials. □

Moreover, there is a bound for the measures μ_m.

Lemma 3.3.10 (logarithmic bound for μ_m)**.** *For all $m \in \mathbb{N}$ and $\lambda \in (0,1)$, we have*

$$\mu_m((0,\lambda)) \leq \frac{n \cdot \log d}{|\log \lambda|}.$$

Proof. This follows in the same way as in the proof of Lück's approximation theorem [Kam19, Proposition 5.18]. A similar argument to prove exactly this argument is also given by Grabowski [Gra15b, Claim A2]. □

This bound then carries over to the spectral measure μ_A.

Lemma 3.3.11 (logarithmic bound for μ_A)**.** *For all $\lambda \in (0,1)$, we have*

$$\mu_A((0,\lambda)) \leq \frac{n \cdot \log d}{|\log \lambda|}.$$

Proof. This can be deduced from the Portmanteau theorem [Els09, Theorem 4.10] as follows: As the subset $(0,\lambda)$ is open, and the measures $(\mu_m)_{m\in\mathbb{N}}$ converge weakly to μ_A, we have

$$\begin{aligned}
\mu_A((0,\lambda)) &= \liminf_{m\to\infty} \mu_m((0,\lambda)) && \text{(Portmanteau Theorem)}\\
&\leq \liminf_{m\to\infty} \frac{n \cdot \log d}{|\log \lambda|} && \text{(Lemma 3.3.10)}\\
&= \frac{n \cdot \log d}{|\log \lambda|}
\end{aligned}$$

□

We can now prove the goal of this section.

Lemma 3.3.12 (spectral measure near zero)**.** *Let G be a finitely generated, sofic group, $n \in \mathbb{N}$ and $A \in M_{n\times n}(\mathbb{Z}G)$ be self-adjoint. Let μ_A be the spectral measure of A. For all $k \in \mathbb{N}_{>2}$, we define*

$$\epsilon_k := \frac{n \cdot \lceil \log d \rceil}{\lfloor \log k \rfloor} \in \mathbb{Q}_{\geq 0}.$$

Then, the sequence $(\epsilon_k)_{k\in\mathbb{N}}$ is computable and tends to zero. Moreover, we have for all $k \in \mathbb{N}_{>2}$ that

$$\mu_A((0,1/k)) \leq \epsilon_k.$$

Proof. The computability of the sequence follows by basic facts about the computability of sequences (see Lemma 2.1.6 and Example 2.1.7). Moreover, we have $\lim_{k\to\infty} \epsilon_k = 0$ and for all $k \in \mathbb{N}_{>2}$, we have by Lemma 3.3.11 that

$$\begin{aligned}
\mu_A((0,1/k)) &\leq \frac{n \cdot \log d}{|\log(1/k)|}\\
&\leq \frac{n \cdot \lceil \log d \rceil}{\lfloor \log k \rfloor}\\
&= \epsilon_k,
\end{aligned}$$

as desired. □

This finishes the proof of Theorem 3.3.1. Note that even the following, slightly stronger statement holds.

Corollary 3.3.13 (the function is computable)**.** *There exists a universal algorithm that effectively computes the ℓ^2-Betti numbers of any sofic group and any matrix over its group ring. More precisely: There exists an algorithm that requires as input*

1. *a finite generating set S of a group G,*
2. *an algorithm solving the word problem in G with respect to S, and*
3. *a matrix $A \in M_{n\times n}(\mathbb{Z}G)$ that is self-adjoint.*

that, assuming that the group G is sofic, outputs a sequence $(q_k)_{k\in\mathbb{N}}$ of rational numbers such that for all $k \in \mathbb{N}$, we have

$$|\dim_{\mathcal{R}G} \ker(\cdot A) - q_k| \leq 2^{-k}.$$

Remark 3.3.14. This is essentially due to the fact that in Lemma 3.3.12, we can bound the constant $||\cdot A||$ from above by the ℓ^1-norm of A, i.e. the sum of all absolute values of coefficients occurring in A, which we can easily calculate from A.

We end this section by some remarks about 'improving' the bounds in question.

Remark 3.3.15. The sequence $\big(n \cdot \log d/ \log k\big)_{k\in\mathbb{N}}$ converges rather 'slowly' to zero. Thus, it is a natural question if we can 'improve' this bound. This is for instance what the Lott-Lück conjecture asks for.

Conjecture 3.3.16 (Lott-Lück, [LL95, Conjecture 7.1]). *Let G be a group and $A \in \mathbb{Z}G$ be self-adjoint. Then, there are $C, \eta > 0$ such that for $\epsilon > 0$ small enough, we have*

$$\mu_A\big((0,\epsilon)\big) \leq C \cdot \epsilon^{\eta},$$

hence, in particular, we have for $k \in \mathbb{N}$ large enough

$$\mu_A\big((0,1/k)\big) \leq \frac{C}{k^{\eta}}.$$

However, this conjecture turned out to be false in general: Grabowski found the following counterexamples.

Theorem 3.3.17 ([Gra15a, Theorem 1.2]). *For every $\delta > 0$, there is a group G_δ and a self-adjoint $S_\delta \in \mathbb{Z}G_\delta$ and a sequence $(\epsilon_i)_{i\in\mathbb{N}}$ of positive real numbers tending to zero such that for all $i \in \mathbb{N}$, we have*

$$\mu_{S_\delta}\big((0,\epsilon_i)\big) > \frac{C}{|\log \epsilon_i|^{1+\delta}}.$$

Grabowski thus conjectures that we cannot 'improve' the bound given in Lemma 3.3.12.

3.3.3 The Determinant Class Conjecture

In this section, we will generalise Theorem 3.3.1 to groups that satisfy the determinant class conjecture. We will first recall the definition of the Fuglede-Kadison determinant and the determinant class conjecture.

Definition 3.3.18 (Fuglede-Kadison determinant [Sch01, Definition 1.3])**.** Let G be a group, $A \in M_{n\times n}(\mathbb{Z}G)$ be self-adjoint and μ_A be the spectral measure of A. Then, we define the *Fuglede-Kadison determinant* of A by

$$\ln\det(A) := \begin{cases} \int_{0+}^{\infty} \log(x)\, d\mu_A(x) & \text{if this integral converges} \\ -\infty & \text{otherwise} \end{cases}$$

where $\int_{0+}^{\infty}$ denotes integration on the set $(0, \infty)$.

Remark 3.3.19. Note that there is only a convergence problem near 0 and no problem for "$x \to \infty$", as μ_A is supported on $[0, ||\cdot A||]$.

Definition 3.3.20 (determinant class [Sch01, Definition 1.4])**.** Let G be a group and $A \in M_{n\times n}(\mathbb{Z}G)$ be self-adjoint. We say that A is *of determinant class* if

$$\ln\det A > -\infty.$$

Conjecture 3.3.21 (determinant class conjecture)**.** *We say that G satisfies the* determinant class conjecture *if every self-adjoint element in $M_{n\times n}(\mathbb{Z}G)$ is of determinant class.*

Example 3.3.22. Sofic groups satisfy the determinant class conjecture. This was proved by Elek and Szabó [ES05, Theorem 5]. For a sofic group G, and $A \in M_{n\times n}(\mathbb{Z}G)$ self-adjoint, we even have $\ln\det A \geq 0$.

From the property of being of determinant class, we can deduce effective computability.

Theorem 3.3.23 ([Gro12, Theorem 6.12])**.** *Let G be a finitely generated group with solvable word problem, and $A \in M_{n\times n}(\mathbb{Z}G)$ be self-adjoint and of determinant class. Then,*

$$\dim_{\mathcal{R}G} \ker(\cdot A) \in \mathrm{EC}_{\geq 0}.$$

Proof. We transform the proof originally given by Groth to fit the context of Section 3.2.

By Corollary 3.2.11, it suffices to find a computable sequence $(\epsilon_k)_{k\in\mathbb{N}}$ of rational numbers such that $\lim_{k\to\infty} \epsilon_k = 0$ and for all $k \in \mathbb{N}_{>2}$, we have

$$\mu_A\Big(\Big(0, \frac{1}{k}\Big)\Big) \leq \epsilon_k.$$

Because A is of determinant class, we have in particular that

$$\int_{0^+}^{1} \log(x)\ d\mu_A(x) > -\infty$$

Let $q \in \mathbb{Q}_{\geq 0}$ such that

$$\int_{0^+}^{1} \log(x)\ d\mu_A(x) \geq -q$$

Then, we have for all $k \in \mathbb{N}_{>0}$:

$$\begin{aligned} -q &\leq \int_{0^+}^{1} \log(x)\ d\mu_A(x) \\ &\leq \int_{0^+}^{(1/k)^-} \log(x)\ d\mu_A(x) && (\log \text{ is negative}) \\ &\leq \log(1/k) \cdot \mu_A((0, 1/k)) && (\text{monotonicity of } \log) \end{aligned}$$

and hence $\mu_A((0, 1/k)) \leq \frac{q}{\log k}$. Hence, such a computable sequence is given by

$$\Big(\epsilon_k := \frac{q}{\lfloor \log k \rfloor}\Big)_{k\in\mathbb{N}},$$

finishing the proof that $\dim_{\mathcal{R}G} \ker(\cdot A) \in \mathrm{EC}_{\geq 0}$. □

Remark 3.3.24. One disadvantage of this proof is that unlike in the sofic case, we cannot (at least a priori) calculate effective bounds $(\epsilon_k)_{k\in\mathbb{N}}$ from G and A.

However, if we even have $\ln \det A \geq 0$, we can achieve the following inequality [Gro12, p. 27].

$$\begin{aligned} q &\leq -\int_{0^+}^{1} \log(x)\ d\mu_A(x) + 1 && (\text{suitable choice of } q) \\ &\leq \int_{1^+}^{d} \log(x)\ d\mu_A(x) + 1 && (\ln \det(A) \geq 0) \\ &\leq \int_{1^+}^{d} x\ d\mu_A(x) + 1 && (\log x \leq x) \\ &\leq \int_{0}^{d} x\ d\mu_A(x) + 1 \\ &= \mathrm{tr}_{\mathcal{R}G}(A) + 1 && (\text{Proposition 1.3.16}) \end{aligned}$$

Since we can compute $\mathrm{tr}_{\mathcal{R}G}(A) + 1$ from A and an algorithm that solves the word problem in G, the same statement as in Corollary 3.3.13 also holds in case that $\ln\det A \geq 0$.

Remark 3.3.25. In the case that one deduces the case for sofic groups from the above above Theorem 3.3.23 (as done by Groth [Gro12]), one essentially makes a detour through the argument given in Chapter 3.3 as the proof by Elek and Szabó [ES05] that sofic groups satisfy the determinant class conjecture, uses similar arguments to the ones presented in that chapter.

We finish this chapter by pointing out two special cases of Theorem 3.3.23.

Corollary 3.3.26. *Let $n \in \mathbb{N}$ and let X be a CW-complex with finite $(n+1)$-skeleton. Let $x_0 \in X$ and $\pi_1(X, x_0)$ be finitely generated and satisfy the determinant class conjecture. Let $\widetilde{X}$ be the universal covering of X. Then, we have*

$$b_n^{(2)}(\pi_1(X, x_0) \curvearrowright \widetilde{X}) \in \mathrm{EC}_{\geq 0}.$$

Proof. By the equivalence discussed in Theorem 1.2.20, we have

$$b_n^{(2)}(\pi_1(X, x_0) \curvearrowright \widetilde{X}) = \dim_{\mathcal{R}G} \ker(\cdot A)$$

for some matrix $A \in M_{n\times n}(\mathbb{Z}G)$, thus Theorem 3.3.23 yields the claim. □

Corollary 3.3.27. *Let $n \in \mathbb{N}$ and let G satisfy the determinant class conjecture and be of finite type F_{n+1}. Then, we have*

$$b_n^{(2)}(G) \in \mathrm{EC}_{\geq 0}.$$

Proof. The group G is finitely generated (Example 1.2.26). Recall from Definition 1.2.27 that

$$b_n^{(2)}(G) = b_n^{(2)}(G \curvearrowright EG_{n+1}),$$

thus, Theorem 1.2.20 and Theorem 3.3.23 yield the claim. □

3.4 Residually Finite Groups

In this chapter, we will focus on finitely presented, residually finite groups (see Definition 1.3.10). We will prove that in this case, all ℓ^2-Betti numbers are effectively computable. Actually, this follows already from Theorem 3.3.1, as residually finite groups are sofic (Example 3.3.5) and finitely presented, residually finite groups have a solvable word problem (see Proposition 3.4.1 below).

However, in this chapter we explore a different proof and use sequences as in Lück's approximation theorem (Theorem 1.3.9) to approximate the ℓ^2-Betti numbers.

3.4.1 Preliminaries: Solvability of the Word Problem

We first prove the statement about the solvability of the word problem.

Proposition 3.4.1. *Let G be a finitely presented, residually finite group. Then, the word problem in G is solvable.*

Proof. Suppose that $G = \langle s_1, \dots, s_j \mid r_1, \dots, r_l \rangle$, where $r_1, \dots, r_l$ are words in $s_1, \dots, s_j$ and their inverses. We need to show that the set

$$R := \{w \in F(S) \mid w = e \text{ in } G\}$$

as well as its complement are semi-decidable.

The set R is semi-decidable: Note that R is the normal subgroup in $F(S)$ generated by $r_1, \dots, r_l$. It is thus equal to the subset of $F(S)$ that arises in finitely many steps from

- the empty word,
- one of the words $r_1, \dots, r_l$,
- taking inverses and products (of two such words)
- conjugating with some s_i or s_i^{-1} (for some $i \in \{1, \dots, j\}$).

Thus, we can enumerate all these words.

The complement of R is semi-decided by Algorithm 4. The first two loops enumerate all group homomorphisms $f : F(S) \to S_p$, given by the images of the generators $s_1, \dots, s_j$, for all $p \in \mathbb{N}$, where S_p is the symmetric group on p elements. Note that in S_p, we can compute all products and inverses explicitly, thus the word problem is solvable. The first if-Statement (Line 4) checks whether f induces a group homomorphism $G \to S_p$. If then, $f(w) \neq e$ in S_p, the algorithm accepts. Otherwise, the algorithm continues to enumerate group homomorphisms $F(S) \to S_p$.

Input : finite presentation $G = \langle s_1, ..., s_j \mid r_1, ..., r_l \rangle$,
$w \in F(S)$

Output: Accept if $w \neq e$ in G, do not halt otherwise

for $p = 1, 2, \ldots$ **do**
  foreach $\sigma_1, \ldots, \sigma_j \in S_p$ **do**
    Set $f := (s_1 \mapsto \sigma_1, \ldots, s_j \mapsto \sigma_j)$
    if $f(r_i) = e$ *in* S_p *for all* $i \in \{1, \ldots, l\}$ **then**
      if $f(w) \neq e$ *in* S_p **then**
        Return 'Yes'
      end
    end
  end
end

Algorithm 4: Semi-deciding $F(S)\backslash R$

The algorithm is indeed correct: If $w = e$ in G, then every group homomorphism $f : G \to S_p$ will satisfy $f(w) = e$, thus the algorithm never halts. If, however, $w \neq e$ in G, because G is residually finite, there exists a $p \in \mathbb{N}$ and a group homomorphism $f : G \to S_p$ such that $f(w) \neq e$. Because the algorithm enumerates all group homomorphisms, eventually this specific one will be found, and the algorithm accepts. □

Remark 3.4.2. This algorithm can be modified to obtain an algorithm of the following type: On input $G = \langle s_1, ..., s_j \mid r_1, ..., r_l \rangle$ and words $w_1, \ldots, w_m \in F(S)$ that do *not* represent the trivial word in G, the output is $p \in \mathbb{N}$ and a group homomorphism $f : G \to S_p$ such that $f(w_i) \neq e$ for all $i \in \{1, \ldots, m\}$.

3.4.2 A Quantitative Version of Lück's Approximation Theorem

We now want to quantify the rate of convergence in Lück's approximation theorem. Recall that its measure-theoretic version can be formulated as follows.

Theorem 1.3.18. *Let X be a free, finite type G-CW complex. Suppose that G is residually finite, and $(G_i)_{i \in \mathbb{N}}$ be a residual chain of G. Let $n \in \mathbb{N}$ and let $\Delta_n \in M_{k \times k}(\mathbb{Z}G)$ denote the n-th Laplacian matrix of the ℓ^2-chain complex of $G \curvearrowright X$. Moreover, for $i \in \mathbb{N}$, let $\pi_i(\Delta_n) \in M_{k \times k}(\mathbb{Z}(G/G_i))$ be the entrywise projection of Δ_n to $\mathbb{Z}(G/G_i)$. Then, $\pi_i(\Delta_n)$ is the n-th Laplacian of the ℓ^2-chain complex of $G/G_i \curvearrowright G_i \backslash X$. Moreover, we have*

$$\mu_{\Delta_n}(\{0\}) = \lim_{i \to \infty} \mu_{\pi_i(\Delta_n)}(\{0\}).$$

In the following, we will work with the algebraic version of ℓ^2-Betti numbers to quantify this convergence. We first define the following notion of trivial intersection.

Definition 3.4.3 (intersects trivially)**.** Let G be a group, $n \in \mathbb{N}$, let $A \in M_{n\times n}(\mathbb{Z}G)$. Let $H \subset G$ be a subgroup of G. We say that A *intersects trivially with* H if every element on the diagonal of A is of the form

$$a_e \cdot e + \sum_{g \in G \setminus H} a_g \cdot g,$$

where $a_g \in \mathbb{Z}$ for all $g \in G\setminus H \cup \{e\}$.

Lemma 3.4.4. *Linear combinations of matrices that intersect trivially with $H \subset G$, intersect trivially with H.*

The main input of this section is the following quantitative version of Lück's approximation theorem.

Theorem 3.4.5 (Lück's approximation theorem, quantitative version)**.** *Let G be a finitely presented, residually finite group, $n \in \mathbb{N}$ and $A \in M_{n\times n}(\mathbb{Z}G)$ be self-adjoint. Let $d \geq 1$ such that the spectral measure μ_A is supported on $[0, d]$. Then, the following holds:*

Let $k \in \mathbb{N}_{\geq 2}$ and $G_k \subset G$ be a normal subgroup of finite index such that $A, A^2, ..., A^{k^2}$ intersect trivially with G_k. Then,

$$|\dim_{\mathcal{R}G} \ker(\cdot A) - \dim_{\mathcal{R}(G/G_k)} \ker(\cdot A_k)| \leq n \cdot \left(1 - \frac{1}{kd}\right)^{k^2} + \frac{n \cdot \log d}{\log k}$$

where we denote by A_k the image of A under the entrywise projection $\mathbb{Z}G \to \mathbb{Z}(G/G_k)$.

Proof. We follow the idea and notation used in the proof of Lück's approximation theorem given by Kammeyer [Kam19, Chapter 5.3]. We denote the spectral measure of A by μ_A (see Definition 1.3.15) and view it as a measure on interval $[0, d]$. Similarly, we denote by μ_k the spectral measure of A_k, which we can view on the same interval $[0, d]$. Recall that by Proposition 1.3.17, we have

$$\mu_A(\{0\}) = \dim_{\mathcal{R}G} \ker(\cdot A),$$
$$\mu_k(\{0\}) = \dim_{\mathcal{R}(G/G_k)} \ker(\cdot A_k).$$

This is where we alter the path of Kammeyer's proof. His next step would have been that the measures μ_k converge weakly to the measure μ_A. This is not good enough for our case. Still, in proving so, we obtain actually the following lemma:

Lemma 3.4.6 ([Kam19, proof of Proposition 5.16]). *Let $p \in \mathbb{R}[T]$ be a polynomial such that $p(A)$ intersects trivially with G_k. Then, we have*

$$\int_0^d p(x)\, d\mu_A(x) = \operatorname{tr}_{\mathcal{R}G}(p(A)) = \operatorname{tr}_{\mathcal{R}(G/G_k)}(p(A_k)) = \int_0^d p(x)\, d\mu_k(x).$$

Proof. The leftmost and the rightmost equality follow from Proposition 1.3.16. The central equality follows from the fact that the trace is the sum of the coefficients belonging to the trivial element $e \in G$. Since $p(A)$ intersects trivially with G_k, these coefficients are not changed by the projection $G \to G/G_k$. ■

Note that this holds in particular for the constant polynomial 1, yielding that $\mu_A([0,d]) = n = \mu_k([0,d])$. The idea is now to exploit this property by approximating the function $\chi_{\{0\}}$ with the polynomial

$$p_k := \left(1 - \frac{x}{d}\right)^{k^2}.$$

For the first calculation, we need the following lemma

Lemma 3.4.7 (logarithmic bound, [Kam19, Proposition 5.18]). *For all $\lambda \in (0,1)$, we have*

$$\mu_k((0,\lambda)) \leq \frac{n \cdot \log d}{|\log \lambda|}.$$

Now, we can calculate the following (plus and minus-signs in the bounds of an integral suggest that we integrate over (half-)open intervals):

$$\begin{aligned}
&\dim_{\mathcal{R}(G/G_k)} \ker(\cdot A_k) \\
&= \mu_k(\{0\}) && \text{(Proposition 1.3.17)} \\
&= \int_0^d p_k(x)\, d\mu_k - \int_{0^+}^d p_k(x)\, d\mu_k \\
&= \int_0^d p_k(x)\, d\mu_A - \int_{0^+}^d p_k(x)\, d\mu_k && \text{(Lemma 3.4.6)} \\
&= \int_0^d p_k(x)\, d\mu_A - \int_{0^+}^{1/k^-} p_k(x)\, d\mu_k - \int_{1/k}^d p_k(x)\, d\mu_k \\
&\geq \int_0^d p_k(x)\, d\mu_A - \mu_k((0,1/k)) - \int_{1/k}^d p_k(x)\, d\mu_k && (p_k(x) \leq 1) \\
&\geq \int_0^d p_k(x)\, d\mu_A - \frac{n \cdot \log d}{k} - \int_{1/k}^d p_k(x)\, d\mu_k && (\text{log bound}, \lambda = 1/k) \\
&\geq \int_0^d p_k(x)\, d\mu_A - \frac{n \cdot \log d}{k} - \int_{1/k}^d p_k(1/k)\, d\mu_k && (p_k(x) \text{ is mon. decreasing})
\end{aligned}$$

$$
\begin{aligned}
&\geq \int_0^d p_k(x)\, d\mu_A - \frac{n \cdot \log d}{k} - \mu_k([1/k, d]) \cdot p_k(1/k) \\
&\geq \int_0^d p_k(x)\, d\mu_A - \frac{n \cdot \log d}{k} - \mu_k([0, d]) \cdot p_k(1/k) && \text{(monotonicity of } \mu_k) \\
&= \int_0^d p_k(x)\, d\mu_A - \frac{n \cdot \log d}{k} - n \cdot \left(1 - \frac{1}{kd}\right)^{k^2} && \text{(Lemma 3.4.6)} \\
&\geq \mu_A(\{0\}) - \frac{n \cdot \log d}{k} - n \cdot \left(1 - \frac{1}{kd}\right)^{k^2} && \text{(monotonicity}, p_k(0) = 1) \\
&= \dim_{\mathcal{R}G} \ker(\cdot A) - \frac{n \cdot \log d}{k} - n \cdot \left(1 - \frac{1}{kd}\right)^{k^2} && \text{(Proposition 1.3.17)}
\end{aligned}
$$

yielding that

$$
\dim_{\mathcal{R}G} \ker(\cdot A) - \dim_{\mathcal{R}(G/G_k)} \ker(\cdot A_k) \leq n \cdot \left(1 - \frac{1}{kd}\right)^{k^2} + \frac{n \cdot \log d}{\log k}
$$

On the other hand, we also have a logarithmic bound for μ_A, i.e.

Lemma 3.4.8 (logarithmic bound, [Lüc94, Theorem 2.3(3)])**.** *For all $\lambda \in (0, 1)$, we have*

$$
\mu_A((0, \lambda)) \leq \frac{n \cdot \log d}{|\log \lambda|}.
$$

Proof. The proof works using the Portmanteau Theorem as in Lemma 3.3.11. ■

Hence, we can perform the same calculation as above, interchanging μ_A and μ_k and obtain

$$
\dim_{\mathcal{R}(G/G_k)} \ker(\cdot A_k) - \dim_{\mathcal{R}G} \ker(\cdot A) \leq n \cdot \left(1 - \frac{1}{kd}\right)^{k^2} + \frac{n \cdot \log d}{\log k}
$$

thus, finishing the proof of Theorem 3.4.5. □

Remark 3.4.9. Note that we can calculate a bound for d from G and A by setting d to be the ℓ^1-norm of A.

3.4.3 Effective Computability

We now want to exploit this quantitative approximation result to obtain that the l^2-Betti numbers in the case of finitely presented, residually finite groups are effectively computable. The idea is to use the characterisation 2 of Proposition 2.2.7. Therefore, we need the following lemma.

Input : finite presentation $\langle s_1, ..., s_j \mid r_1, ..., r_l \rangle$,
$n \in \mathbb{N}$,
$A \in M_{n\times n}(\mathbb{Z}G)$,
$k \in \mathbb{N}$
Output: $\dim_{\mathcal{R}(G/G_k)} \ker(\cdot A_k)$

calculate $A, A^2, \dots A^{k^2}$
`/* collect non-trivial elements */`
initialise empty list nontriv_elts
foreach *group element* g *in the diagonals of* $A, A^2, \dots A^{k^2}$ **do**
  Run W on g
  if $g \neq e$ *in* G **then**
    Add g to nontriv_elts
  end
end
`/* find suitable subgroup */`
Set $f := T$(nontriv_elts)
`/* Now, have found` $G_k := \ker f$ `*/`
Enumerate elements of subgroup H, generated by $f(s_1), \dots, f(s_j)$
Enumerate composition table of H
Define $A_k := f(A) \in M_{n\times n}(\mathbb{Z}H)$
Rewrite A_k as matrix in $M_{n|H|\times n|H|}(\mathbb{Q})$
Calculate $\dim_Q \ker A_k$ (using the Gauß algorithm)
Return $(\dim_Q \ker A_k)/|H|$

Algorithm 5: Computing $\dim_{\mathcal{R}(G/G_k)} \ker(\cdot A_k)$

Lemma 3.4.10. *Let G be a finitely presented, residually finite group, given by a finite presentation $\langle s_1, ..., s_j \mid r_1, ..., r_l \rangle$. Let $n \in \mathbb{N}$ and $A \in M_{n\times n}(\mathbb{Z}G)$, where each entry is given by a linear combination of words in $s_1, ..., s_j$ and its inverses.*

Then, there is an algorithm that, on input $k \in \mathbb{N}$, determines a normal subgroup $G_k \subset G$ of finite index such that $A, A^2, \dots A^{k^2}$ intersect trivially with G_k and has as output

$$\dim_{\mathcal{R}(G/G_k)} \ker(\cdot A_k).$$

Proof. Let W be an algorithm that solves the word problem in G (see Proposition 3.4.1). Let T be an algorithm, as in Remark 3.4.2, i.e. that given a list $w_1, \dots, w_m \in F(S)$ of words *not* representing the trivial element, outputs a group homomorphism $f : G \to S_p$ such that $f(w_i) \neq e$ for all $i \in \{1, \dots, m\}$. The desired solution is then given by Algorithm 5.

To conclude, we need to prove the following for this algorithm:

1. For $G_k := \ker f$, we have that G_k is a finite index, normal subgroup and

$A, A^2, \dots, A^{k^2}$ intersect trivially with G_k.

2. The algorithm indeed returns $\dim_{\mathcal{R}(G/G_k)} \ker(\cdot A_k)$.

Ad 1) The subgroup $G_k := \ker f$ is the kernel of a group homomorphism to a finite group, hence G_k is normal and of finite index. The matrices $A, A^2, \dots, A^{k^2}$ intersect trivially with G_k by construction of the algorithm, since the non-trivial elements are given as input to the algorithm T.

Ad 2) We have $G/G_k \cong \operatorname{im} f =: H$. Because H is a finite group, its von Neumann algebra is isomorphic to $\mathbb{C}[H]$ (Example 1.1.11) and we have

$$\begin{aligned}\dim_{\mathcal{R}(G/G_k)} \ker A_k &= \dim_{\mathcal{R}H} \ker A_k \\ &= \frac{\dim_{\mathbb{C}} \ker A_k}{|H|} \\ &= \frac{\dim_{\mathbb{Q}} \ker A_k}{|H|}\end{aligned}$$

which is exactly the output of the algorithm. □

Finally, we can prove the following theorem.

Theorem 3.4.11 (effective computability)**.** *Let G be a finitely presented, residually finite group, $n \in \mathbb{N}$ and $A \in M_{n\times n}(\mathbb{Z}G)$. Then, we have*

$$\dim_{\mathcal{R}G} \ker(\cdot A) \in \mathrm{EC}_{\geq 0}.$$

Proof. By Lemma 3.4.10, there is a sequence $(G_k)_{k\in\mathbb{N}}$ of normal subgroups of finite index such that for all $k \in \mathbb{N}$, the matrices $A, A^2, \dots, A^{k^2}$ intersect trivially with G_k and such that

$$\big(\dim_{\mathcal{R}(G/G_k)} \ker(\cdot A_k)\big)_{k\in\mathbb{N}}$$

is a computable sequence (where again $A_k \in \mathbb{Z}(G/G_k)$ is the canonical projection of A to $\mathbb{Z}(G/G_k)$). Moreover, by the quantitative version of Lück's approximation theorem (Theorem 3.4.5), we have for all $k \in \mathbb{N}_{>2}$,

$$\begin{aligned}|\dim_{\mathcal{R}G} \ker(\cdot A) - \dim_{\mathcal{R}(G/G_k)} \ker(\cdot A_k)| &\leq n \cdot \Big(1 - \frac{1}{kd}\Big)^{k^2} + \frac{n \cdot \log d}{\log k} \\ &\leq n \cdot \Big(1 - \frac{1}{kd}\Big)^{k^2} + \frac{n \cdot \lceil \log d \rceil}{\lfloor \log k \rfloor}\end{aligned}$$

By basic facts about the computability of sequences (see Lemma 2.1.6 and Example 2.1.7), the sequence on the right hand side is computable. By elementary calculus, this sequence converges to zero for $k \to \infty$. Hence, Proposition 2.2.7 implies that $\dim_{\mathcal{R}G} \ker(\cdot A)$ is effectively computable. □

Remark 3.4.12. All the steps used in these proofs are constructive, i.e. in theory, we could give an explicit algorithm that, given a presentation for G, $n \in \mathbb{N}$ and $A \in M_{n\times n}(\mathbb{Z}G)$, outputs such a sequence $\left(\dim_{\mathcal{R}(G/G_k)} \ker(\cdot A_k)\right)_{k\in\mathbb{N}}$ and the sequence on the right-hand side of the inequality.

Remark 3.4.13. Given a computational model of a finite CW-complex X whose fundamental group is finitely presented, we can algorithmically determine the combinatorial Laplacians of the $\pi_1(X, x_0)$-CW complex $\widetilde{X}$ (where $x_0 \in X$ is a point of the 0-skeleton). Thus, also in this 'topological' setting, the obtain the slightly stronger statement that the ℓ^2-Betti numbers are effectively computed by a universal algorithm.

4 Implementation in Lean

4.1 What is the Lean Theorem Prover?

The *Lean Theorem Prover* is used to *formally verify* mathematical theorems. *Formal verification* describes the process of establishing claims about precise statements using computational methods. We distinguish between two main types:

1. *Automated Theorem Proving* describes methods where a proof is found (at least to a certain extent) automatically by a computer. This can for instance be done for formulas in propositional logic (where enumerating all possibilities in a truth table suffices to prove a claim). Moreover, *Computer Algebra Systems* fall into this category. These are systems that focus on explicit computations and by finding these provide some kind of proof for the result (at least, assuming that these systems work precisely enough).
2. On the other hand, *Interactive Theorem Proving* (sometimes also called *Proof assistants*) focusses on verifying proofs, i.e. the user has to input a statement and a proof in a formal language and the program then checks the proof and outputs whether it is valid or not. Note that this requires a high level of precision: Because all theorems are to be deduced from a small set of axioms, every statement in the proof has to be justified using the implemented axioms and the claims proven before. This implies that proofs tend to be much longer and far more detailed than their counterparts in 'natural' language.

The Lean Theorem Prover aims to "bridge this gap" between automated and interactive theorem proving [AMK21, Chapter 1.1]. However, from a beginner's point of view, it should be noted that the 'automated' part (at least in the current version) has restricted capabilities: Some types of statements can be proved automatically and this often helps to speed up and simplify proofs. One should however not expect Lean to be able to prove more complex theorems entirely without user input. In general, one has to write a proof where one can use some tricks to simplify this process.

A critical reader might now ask: Why should we use such a program if, nonetheless, we have to write the proof? The most obvious reason is that the computer program assures us of the correctness of the proof. After all, humans are quite prone to error and sometimes, even published proofs turn out to be wrong because a tiny detail was overlooked. A theorem prover forces the user to make these details explicit and checks them. This might give the user a better understanding of the details and new insights about the proof. Finally, if afterwards, we come up with a generalisation or similar statement that uses the 'same' proof, a theorem prover can point exactly to the steps where the proof fails and where we thus have to modify the argument.

In this thesis, we will focus on Lean 3. The Lean Theorem Prover uses *dependent type theory* as the language for its foundations. In contrast to set theory, type theory is often better suited for computational systems. In short, the name stems from the fact that every expression has an associated *type*. For instance, $2 + 1$ may have the type $\mathbb{N}$, whereas f may denote a function of type $\mathbb{N} \to \mathbb{N}$. In particular, type theory is better suited than set theory to help us keep track of the types of objects that we are working with.

One good reason for choosing Lean 3 is the `mathlib`, a formal library that establishes many facts from undergraduate mathematics. It is built as a basis for research level mathematics. For more details on the `mathlib`, see an article by the mathlib community [Com20]. For an introduction into theorem proving in Lean, see [AMK21].

In the following sections, we will discuss a Lean project written by the author of this thesis (see Section 4.2 on how to obtain the source files). To aid beginners in Lean, we end this section with a few practical remarks that will help get an overview on the project.

The general structure of a theorem is the following:

```
theorem name (h : hypothesis1) : claim
:= begin
 .... ,
 .... ,
end
```

A theorem starts with the word `theorem` (alternatively, we can also use `lemma`), followed by the name of the theorem. Then, the hypotheses of this statement are listed, each in parentheses. On the left of the colon, we have the name of the hypothesis (which can be used to reference this hypothesis in the proof), on the right, we find the statement of the hypothesis. There can also be multiple hypotheses, or even none. Then, there is a colon, after which we find the claim of the theorem. Finally, the proof of the theorem is enclosed by `begin` and `end`. There are also different ways to give a proof in Lean without using a `begin-end` block but for our purposes, entering this *tactic mode* will often be convenient. Note that every line in such a block ends with a comma.

Consider the following easy example.

```
theorem and_comm' {p q: Prop} (h : p ∧ q) : q ∧ p
:= begin
  cases h with hp hq,
```

```
  exact and.intro hq hp,
end
```

The theorem `and_comm'` states that for propositions p and q that satisfy $p \wedge q$, we have $q \wedge p$. In the proof, the first line splits the assumption `h` into the assumptions `(hp: p)` and `(hq: q)`. The goal is then solved by the lemma `and.intro` provided with these two facts.

If you are using `vscode`, you can inspect the current state of the proof by placing your cursor inside the proof. In the standard view, the current goals are then shown on the right hand side of the window. In the above example, we find the following goal after the `begin` command

```
p q: Prop
h: p ∧ q
 ⊢ q ∧ p
```

where the symbol ⊢ separates the hypotheses from the claim. As we move downwards, the goal(s) change to

```
p q: Prop
hp: p
hq: q
 ⊢ q ∧ p
```

and one line below, we reach the message `goals accomplished`, indicating that the proof is finished.

Outside of theorems, we can use the `#check` command to check the type of a theorem. For instance, we can use

```
#check and_comm'
```

to check the theorem that we have just proved or

```
#check and.intro
```

to check the lemma that the theorem references. For the latter, in `vscode`, we can also hover the expression `and.intro` in the proof to get this information.

4.2 Overview of the Implementation

In this section, we give an overview of the Lean project of this master's thesis. The files can be found on the attached DVD. Alternatively, you can find them in the `git` repository at `https://git.uni-regensburg.de/usm34387/lean-master-thesis-release.git`. To access, type into a terminal

```
git clone https://git.uni-regensburg.de/usm34387/lean-master-thesis-release.
    git
```

and then follow the instructions provided in `installation-guidelines.txt`.

4.2.1 List of files

The project consists of the following files (in alphabetic order) that are contained in the `src/` folder. The dependencies between the files are visualised in Figure 4.1.

`computable_sequences.lean` provides an inductive definition of computability of sequences. It also contains the definition of the computability classes EC, LC and RC (Definition 2.2.2) as well as some basic properties thereof, especially the equality $\mathrm{EC} = \mathrm{LC} \cap \mathrm{RC}$ (Proposition 2.2.4).

`determinant_class_EC.lean:` The main goal is to prove that ℓ^2-Betti numbers arising from groups with solvable word problem that satisfy the determinant class conjecture are effectively computable (Theorem 3.3.23). The file mimics the proof of Theorem 3.3.23, using the results from `general_bound_LC.lean`.

`general_bound_LC.lean` formalises the implication that 1 implies 2 in Theorem 3.2.9, i.e. if a group has a solvable word problem and admits a computable bound on its spectral measure near zero, then its ℓ^2-Betti numbers are left-computable. As a consequence from `general_RC.lean`, they are also right-computable, thus effectively computable.

`general_lemmas.lean` provides technical lemmas needed for the files `general_bound_LC.lean` and `general_RC.lean`.

`general_RC.lean` formalises the result that the solvability of the word problem implies right-computability of the ℓ^2-Betti numbers (Theorem 3.2.1).

`limit_lemmas.lean` provides some basic statements about the convergence of real sequences that are usually treated in introductory courses on calculus.

`Lueck_approx_EC.lean` formalises Theorem 3.4.11 that deduces effective computability of the ℓ^2-Betti numbers from the quantitative version of Lück's approximation theorem (Theorem 3.4.5).

`quantitative_Lueck_lemmas.lean` provides lemmas for `quantitative_Lueck.lean`.

`quantitative_Lueck.lean` formalises the quantitative version of Lück's approximation theorem (Theorem 3.4.5).

`spectral_measure.lean` defines the notions of a *traced algebra* (i.e. an algebra with a map to its base ring) and spectral measures (Definition 1.3.15).

4.2.2 Some Specific Definitions

In the implementation, some definitions do not match the notions defined in the theoretical part of this thesis. As a general rule of thumb, it is often faster to use some characteristic property or defining equality that is 'close' to the necessary property of this object in the implementation. Compared to the usual definitions, this spares us the implementation of large parts of basic statements of the theory. Often, these basics would take up a large part of the implementation yet not being directly related to the main results. That is why we take the faster approach.

In the following, we will explain some of the definitions and how these variants are related to the situations in this thesis.

- The notion of a `traced_algebra` (over a ring R) is defined as an R-algebra A together with a map $\mathrm{tr} : A \to R$. At first glance, this seems insufficient, as we would expect tr to satisfy some properties, e.g. R-linearity or the trace property. But for our needs, the mere existence of the trace map is enough, and thus we do not make this definition more complicated than absolutely necessary.

```
class traced_algebra (R: Type*) (A: Type*)
[comm_semiring R] [semiring A]  extends algebra R A:=
  (tr: A →  R)
end traced_algebra
```

Code 4.1: spectral_measure.lean

- We then define `is_spectral_measure` as in Code 4.2. This definition uses the characterisation of Proposition 1.3.16, i.e. μ is the spectral measure of a if for all polynomials $p \in \mathbb{R}[X]$, we have

$$\int_0^d p(x)\ d\mu(x) = \mathrm{tr}(p(a)).$$

  Note that in the data of the definition, we include the constant d.

- `computable_seq` is defined using an inductive definition (see Code 4.3), i.e. we obtain the 'smallest' such notion satisfying all of these properties. This is supposed to mean that a sequence is computable (Definition 2.2.1). Note that the notion of `computable_seq` might be 'narrower' than the definition given in

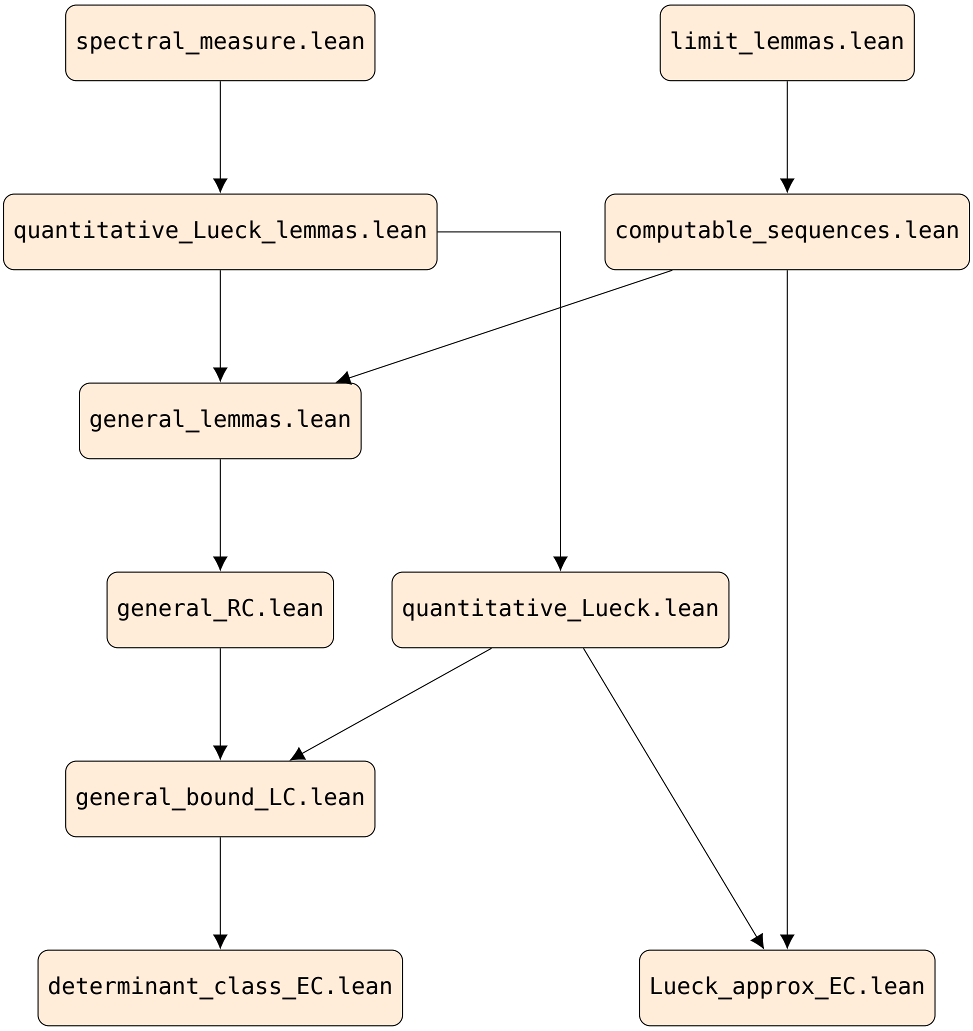

Figure 4.1: dependencies between the files in the Lean project

```
universe u
variables  (A: Type u)
[ring A] [algebra real A] [traced_algebra real A]

def is_spectral_measure (a : A)
  ( μ : measure ℝ) (d : ℝ)
    :Prop
  := ∀(p : polynomial ℝ),
      ∫(x : ℝ) in Icc 0 d,
        ( λ (x : ℝ), polynomial.eval x p) x ∂μ
      = traced_algebra.tr
      (polynomial.eval₂ algebra.to_ring_hom a p)
      --  integral of p(x) from 0 to d = tr (p(a))
```

Code 4.2: spectral_measure.lean

Definition 2.2.1. Some of the claimed properties are shown in Lemma 2.1.6 and Example 2.1.7.

- The definition of EC, LC, and RC are given in Code 4.4. These notions are formulated as in characterisations of theses computability classes (Proposition 2.2.7 and Proposition 2.2.9). In Lean, we write `EC x` for some $x \in \mathbb{R}$ to denote that x is effectively computable.
- Finally, one should keep in mind that limits of sequences are defined using filters in Lean.

```
def seq_limit (a: ℕ → ℝ) (l: ℝ)
    := filter.tendsto a filter.at_top (nhds l)
```

Code 4.5: limit_lemmas.lean

However, we will often use a lemma stating that this is equivalent to the 'classical' definition.

```
lemma seq_limit_classical {a: ℕ → ℝ} {l: ℝ} :
      seq_limit a l ↔ ∀ ε > 0, ∃ N, ∀ n ≥ N, |a n - l| < ε
```

Code 4.6: limit_lemmas.lean

4.2.3 Main Theorems

We will explain the four main theorems of the project and their hypotheses.

```
inductive computable_seq : ( ℕ → ℝ) → Prop
 | inclusion_computable: computable_seq ( λ n, (n: ℝ))
 | constant_computable: ∀q: ℚ, computable_seq ( λ n, q)
 | sum_computable: ∀{a b: ℕ → ℝ}, computable_seq a →
              computable_seq b → computable_seq (a+b)
 | product_computable: ∀{a b: ℕ → ℝ}, computable_seq a →
              computable_seq b → computable_seq ( λ n, (a n)*(b n))
 | div_computable: ∀{a b: ℕ → ℝ}, computable_seq a →
              computable_seq b → computable_seq ( λ n, (a n)/(b n))
 | pow_computable: ∀{a b: ℕ → ℝ}, computable_seq a →
              computable_seq b → computable_seq ( λ n, (a n)^(b n))
 | floor_computable : ∀{a: ℕ → ℝ}, computable_seq a →
                    computable_seq ( λ n, floor(a n))
 | log_computable : computable_seq ( λ n, log n)
 | case_distinction_computable: ∀{a b: ℕ → ℝ} (N: ℕ), computable_seq a →
                  computable_seq b →
                 computable_seq ( λ n, ite (n ≤ N) (a n) (b n))
 | max_computable : ∀{a b: ℕ → ℝ} (N: ℕ), computable_seq a →
                  computable_seq b →
                 computable_seq ( λ n, max (a n) (b n))
 | composition_computable : ∀ {a b: ℕ → ℝ}, computable_seq a →
                  computable_seq b →
                  computable_seq ( λ n, a ((floor (b n)).to_nat))
```

Code 4.3: computable_sequences.lean

```
-- effective computability: we can estimate the error
def EC (x: ℝ) : Prop
    := ∃ (q ε : ℕ → ℝ), computable_seq q ∧ computable_seq ε
                        ∧ seq_limit ε 0
                   ∧ ( ∀ n, |x - (q n)| ≤ ε n)

--left computability: there is a computable sequence
-- converging from below
def LC (x: ℝ) : Prop
    := ∃ (q : ℕ → ℝ), computable_seq q
                    ∧ seq_limit q x
                    ∧ ( ∀ n, q n ≤ x)

-- and analogously: right computability with a sequence from above
def RC (x: ℝ) : Prop
    := ∃ (q : ℕ → ℝ), computable_seq q
                    ∧ seq_limit q x
                    ∧ ( ∀ n, q n ≥ x)
```

Code 4.4: computable_sequences.lean

General Right Computability We first discuss the following implementation of Theorem 3.2.1.

```
theorem general_RC : RC ( μ {0}).to_real
```

Code 4.7: general_RC.lean

The theorem is named `general_RC` and states that $\mu\ \{0\}$ is right-computable. But, because for general measures, $\mu\ \{0\}$ can take values in $\mathbb{R}_{\geq 0} \cup \{+\infty\}$, in Lean, $\mu\ \{0\}$ is of type `ennreal`. In order to cast this number to a real number, we thus have to write `.to_real`.

Obviously, we need some hypotheses for this theorem that we will explain in the following. The following lines state that RG is a traced algebra over $\mathbb{R}$. In the situation of Theorem 3.2.1, this corresponds to the $\mathbb{R}$-algebra $M_{n\times n}(\mathbb{R}G)$ (we already use the group ring over $\mathbb{R}$ to avoid casting later) together with its trace to $\mathbb{R}$.

```
universe u
variables
{RG: Type u} [ringRG: ring RG]
[algebra ℝ RG]
[tr_alg: traced_algebra ℝ RG] --RG is an algebra with trace over R
```

Code 4.8: general_RC.lean

Moreover, we need an element of the algebra RG. Because Lean uses type theory (instead of set theory) as a foundation of mathematics, we have to write

```
(A: RG) -- A is an element of RG
```

Code 4.9: general_RC.lean

Next, we know that A has a spectral measure. In the implementation, we assume the we have a finite measure μ that, on $[0, d]$ is a spectral measure of A.

```
{d: ℚ} (d_ge1: (d: ℝ)≥ 1)
( μ: measure ℝ) [finite_measure μ]
( μ_spec_meas: is_spectral_measure RG A μ d)
```

Code 4.10: general_RC.lean

We denote for all k by `pk k` the polynomial

$$\left(1 - \frac{1}{d} \cdot X\right)^k \in \mathbb{R}[X].$$

```
local notation ‘pk‘ k  :=  (polynomial.C 1- polynomial.C(1/(d: ℝ)) *
    polynomial.X)^k
```

Code 4.11: general_RC.lean

Moreover, we assume that the sequence

$$k \mapsto \operatorname{tr}\big((\mathrm{pk}\ k)(A)\big)$$

is computable. Compared to Theorem 3.2.1, this is justified because we assume the solvability of the word problem, hence, we can produce an algorithm computing this sequence.

```
variables
(traces_computable: computable_seq
                        ( λ k, traced_algebra.tr
                        (polynomial.eval₂ algebra.to_ring_hom A (pk k))))
```

Code 4.12: general_RC.lean

General Left Computability Next, we will discuss an implementation of Theorem 3.2.9, stating that $\mu\,\{0\}$ (after casting to $\mathbb{R}$) is left-computable.

```
theorem bound_implies_LC : LC μ( {0}).to_real
```

Code 4.13: general_bound_LC.lean

In addition to the hypotheses discussed above for `general_RC.lean`, we need the following assumptions.

We have a rational number $t \in \mathbb{Q}$ such that $\mu((0,d]) \leq t$. This is justified by the fact that the measure is finite. We would also obtain $\mu(\top) < \infty$ from this finiteness condition, where $\mu(\top)$ denotes the total measure of μ. However, in the end, this additional assumption makes the proof a bit shorter.

```
variables {t: ℚ}
( μ_tot: ennreal.to_real( μ (Ioc 0 d)) ≤ t)
```

Code 4.14: general_bound_LC.lean

In the file `general_bound_LC.lean`, we define the polynomials slightly differently than in `general_RC.lean` and we assume again that a sequence of traces is a computable sequence. Again, this stems from the fact that the word problem is solvable.

```
local notation ‘pkk‘ k  :=  (polynomial.C 1- polynomial.C(1/(d: ℝ)) *
    polynomial.X)^(k*k)

/-the sequence tr (pkk (A)) is computable-/
variables
(traces_computable: computable_seq
                        ( λ k, traced_algebra.tr
                        (polynomial.eval₂ algebra.to_ring_hom A (pkk k))))
```

Code 4.15: general_bound_LC.lean

Finally, we have a sequence, called `bound`, that is computable, tends to zero and satisfies for all $n \in \mathbb{N}_{>0}$ that $\mu((0,1/n)) \leq \text{bound}(n)$, hence it plays the role of the sequence $(\epsilon_n)_{n\in\mathbb{N}}$ in Theorem 3.2.9.

Note that we can also demand extend the condition to all $n \in \mathbb{N}$, because in Lean, we have $0^{-1} = 0$.

```
{bound: ℕ → ℝ}
(bound_computable: computable_seq bound)
(bound_to_zero: seq_limit bound (0: ℝ ))
(bound_bound: ∀(n: ℕ), ( μ (Ioc 0 (n: ℝ)⁻¹)).to_real
                              ≤ bound n)
```

Code 4.16: general_bound_LC.lean

Finally, the file `general_bound_LC.lean` contains the obvious corollary stating effective computability.

```
theorem bound_implies_EC : EC ( μ {0}).to_real
```

Code 4.17: general_bound_LC.lean

This theorem follows from `general_RC` and `bound_implies_LC` using the following lemma.

```
lemma EC_iff_LC_RC {x: ℝ} : EC x ↔ LC x ∧ RC x
```

Code 4.18: computable_sequences.lean

Effective Computability from the determinant class conjecture The third main theorem implements Theorem 3.3.23.

```
theorem det_class_implies_EC : EC ( μ {0}).to_real
```

Code 4.19: determinant_class_EC.lean

Also in this case, we have some hypotheses that have been discussed before.

```
universe u
variables
{RG: Type u} [ringRG: ring RG]
[algebra ℝ RG]
[tr_alg: traced_algebra ℝ RG] --RG is an algebra with trace over R
(A: RG) -- A is an element of RG
        -- actually of ZG, but we cannot model this here (this will, however be
        -- reflected in the logarithmic bound)
{d: ℚ} (d_ge1: (d: ℝ)≥ 1)
( μ: measure ℝ) [finite_measure μ]
( μ_spec_meas: is_spectral_measure RG A μ d)
--μ has a total measure
{t: ℚ} (t_pos: t ≥ 0)
( μ_tot: ennreal.to_real( μ (Ioc 0 d)) ≤ t)
```

Code 4.20: determinant_class_EC.lean

We define two sequences of polynomials whose traces are computable. Again, these facts resemble the assumption that the word problem is solvable. Note that actually, we could deduce the first variable from the second (as this is the sequence with k^2 inserted instead of k) but for convenience, we state the two assumptions.

```
local notation ‘pkk‘ k  :=  (polynomial.C 1- polynomial.C(1/(d: ℝ)) *
    polynomial.X)^(k*k)
local notation ‘pk‘ k  :=  (polynomial.C 1- polynomial.C(1/(d: ℝ)) *
    polynomial.X)^k

/-the sequences tr (pk (A)) and tr (pkk (A)) are computable-/
variables
```

```
(traces_computable: computable_seq
                        ( λ k, traced_algebra.tr
                        (polynomial.eval₂ algebra.to_ring_hom A (pkk k))))
(traces_computable₂: computable_seq
                        ( λ k, traced_algebra.tr
                        (polynomial.eval₂ algebra.to_ring_hom A (pk k))))
```

Code 4.21: determinant_class_EC.lean

Finally, we have the assumption on the determinant class conjecture. In the implementation, this just takes the form that the log function is integrable with respect to the measure μ.

```
variable (determinant_class : integrable log μ)
```

Code 4.22: determinant_class_EC.lean

Effective Computability from Lück's approximation theorem Finally, there is an implementation of Theorem 3.4.11

```
theorem EC_of_Lueck : EC ( μ {0}).to_real
```

Code 4.23: Lueck_approx_EC.lean

with the following hypotheses: The first few lines have already been discussed above

```
universe u
variables
{RG: Type u} [ringRG: ring RG]
[tr_alg: traced_algebra ℝ RG] --RG is an algebra with trace over R
(A: RG) -- A is an element of RG
        -- actually of ZG, but we cannot model this here (this will, however be
        -- reflected in the logarithmic bound)
{d: ℚ} (d_ge1: (d: ℝ) ≥ 1)
( μ: measure ℝ) [finite_measure μ]
( μ_spec_meas: is_spectral_measure RG A μ d)
```

Code 4.24: Lueck_approx_EC.lean

Next, we assume that a logarithmic bound for μ exists (Lemma 3.4.8). Note that here, it is formulated for the half-open interval $(0, 1/k]$ instead of $(0, 1/k)$, but this does not make a qualitative difference.

```
{c: ℚ}
( μ_log_bound: ∀(k: ℕ), k ≥ 2 → ennreal.to_real ( μ (Ioc 0 (k: ℝ)⁻¹)) ≤ (c / (log k)))
```

Code 4.25: Lueck_approx_EC.lean

Again, we have a bound on the value $\mu((0,d])$.

```
{t: ℚ}
( μ_tot: ennreal.to_real( μ (Ioc 0 d)) ≤ t)
```

Code 4.26: Lueck_approx_EC.lean

We have a family of the quotient rings $(\mathbb{R}(G/G_k))_{k\in\mathbb{N}}$ and measures $(\mu_k)_{k\in\mathbb{N}}$ (with the notation in Theorem 3.4.11).

```
(RGquot: ℕ → Type u) [quot_ring: Π n, ring (RGquot n)]
[tr_alg_quot : Π n, traced_algebra ℝ (RGquot n)]
( μquot: ℕ → measure ℝ) [ ∀ n, finite_measure μ(quot n)]
--μquot n also has a total measure
( μquot_tot: ∀ n, ennreal.to_real(( μquot n) (Ioc 0 d)) ≤ t)
```

Code 4.27: Lueck_approx_EC.lean

We then have the projections $\mathbb{R}G \to \mathbb{R}(G/G_k)$ and denote `Aquot k` the image of A under this projection.

```
(proj: Π  n: ℕ , RG → RGquot n)

-- we can define Aquot, which is the projection of A into the quotient
local notation ‘Aquot‘ n := (proj n) A
```

Code 4.28: Lueck_approx_EC.lean

Moreover, these ‘quotient’ measures are also spectral measures and satisfy a logarithmic bound with the same constant c (Lemma 3.4.7).

```
variables
-- now, μquot are spectral measures for Aquot
( μquot_spec_meas: ∀n, is_spectral_measure (RGquot n) (Aquot n) ( μquot n) d )
-- also this measure has a logarithmic bound
( μquot_log_bound: ∀(n: ℕ) (k: ℕ), k ≥ 2 → ennreal.to_real (( μquot n) (Ioc 0 (k: ℝ )⁻¹)) ≤ (c / (log k)))
```

Code 4.29: Lueck_approx_EC.lean

We have that $\mathrm{tr}(p(A)) = \mathrm{tr}(p(A_k))$ (where A_k is the projection of A) if p is a polynomial with $\deg p \leq k^2$ (see Lemma 3.4.6).

```
(tr_coincide: ∀ (n: ℕ)  (p: polynomial ℝ) , polynomial.nat_degree p ≤ n*n →
              traced_algebra.tr (polynomial.eval₂ algebra.to_ring_hom A p)
            = @traced_algebra.tr ℝ (RGquot n) _ _ (tr_alg_quot n)
            (polynomial.eval₂ algebra.to_ring_hom (Aquot n) p))
```

Code 4.30: Lueck_approx_EC.lean

Finally, the sequence $(\mu_n \{0\})_{n \in \mathbb{N}}$ is computable (Lemma 3.4.10).

```
variable
(kernels_computable: computable_seq ( λ n, (( μquot n) {0}).to_real))
```

Code 4.31: Lueck_approx_EC.lean

```
theorem general_RC : RC ( μ {0}).to_real
:= begin
    unfold RC,
    let q: ℕ → ℝ := ( λ k, @traced_algebra.tr ℝ RG _ _ tr_alg
                        (polynomial.eval₂ algebra.to_ring_hom A (pk k))),
    use q,
    split,
        exact traces_computable,
    split,
        exact tr_seq_converges A μ_spec_meas d_ge1,
        exact seq_is_above A μ μ_spec_meas d_ge1,
end
```

Code 4.32: general_RC.lean

4.3 Case Study: General Right-Computability

In this section, we take a closer look at the theorem `general_RC` in the file `general_RC.lean` and discuss some of the aspects of its proof.

4.3.1 The Main Theorem

The theorem `general_RC` (without hypotheses) and its proof are given in Code 4.32.

In the first line, we find the statement of the theorem. Between `begin` and `end`, the proof of this theorem is enclosed. We begin by unfolding the definition of `RC`, and obtain the goal

```
∃(q : ℕ → ℝ),
computable_seq q ∧ seq_limit q ( ⇑μ {0}).to_real ∧ ∀ (n : ℕ), q n ≥ ( ⇑μ
    {0}).to_real
```

In the proof, we then define q to be the sequence $\left(\mathrm{tr}((\mathrm{pk}\ k)(A))\right)_{k\in\mathbb{N}}$ and by `use q`, we ask Lean to use this sequence for the existential proof. We use `split` to split the resulting conjunction into its parts. The first one is exactly the assumption `traces_computable` whereas the other two conjuncts are treated by the lemmas `tr_seq_converges` and `seq_is_above`, respectively.

4.3.2 The Lemma `seq_is_above`

Let us take a closer look on the lemma `seq_is_above` (Code 4.33).

The proof starts with `intro k`, which fixes $k \in \mathbb{N}$, thus eliminating the all quantifier

```
lemma seq_is_above ( μ_spec_meas: is_spectral_measure RG A μ d)
(dge1: (d: ℝ)≥ 1)
: ∀ (k: ℕ), @traced_algebra.tr ℝ RG _ _ tr_alg
                        (polynomial.eval₂ algebra.to_ring_hom A (pk k))
                        ≥ ( μ {0}).to_real
:= begin
    intro k,
    rw ← μ_spec_meas (pk k),
    simp,
    have dnonneg : (d: ℝ) ≥ 0  := d_nonneg dge1,
    rw int_Icc_vs_Ioc dnonneg μ (pkk_function_is_integrable_Icc μ k (d: ℝ)),
    simp,
    exact interval_integral.integral_nonneg dnonneg
        (pkk_nonneg k dge1),
end
```

Code 4.33: general_RC.lean

in the goal. We then use the `rw` (rewrite) command to rewrite the claim using μ`_spec_meas`, i.e. the fact that μ is a spectral measure of A. We simplify using `simp` and obtain the following claim.

```
↑μ( {0}).to_real
≤ ∫ (x : ℝ) in Icc 0 ↑d, (1 - ↑(d)⁻¹ * x) ^ k ∂μ
```

At this point, we rewrite using the lemma `int_Icc_vs_Ioc` (from `quantitative_Lueck_lemmas.lean`) which states the following: For a measure μ, an integrable function f on $[a, b]$ with $a \leq b$, we have that

$$\int_{[a,b]} f \, d\mu = f(a) \cdot \mu(\{a\}) + \int_{a+}^{b} f \, d\mu,$$

where $\int_{a+}^{b}$ denotes integration over the interval $(a, b]$.

```
lemma int_Icc_vs_Ioc {a b: ℝ} (hab: a ≤ b)
  ( μ: measure ℝ) [locally_finite_measure μ]
  {f: ℝ → ℝ}
  (fint: measure_theory.integrable_on f (Icc a b) μ):
  ∫(x: ℝ) in Icc a b, f x ∂μ = (f a)* μ( {a}).to_real + ∫ x in a..b, f x ∂μ
```

Code 4.34: quantitative_Lueck_lemmas.lean

Back in the proof of `seq_is_above`, we simplify again to obtain the following claim,

```
0 ≤ ∫ (x : ℝ) in ↑0..d, (1 - ↑(d)⁻¹ * x) ^ k ∂μ
```

which is then solved by the theorem `interval_integral.integral_nonneg` (stating that integrals of nonnegative functions are nonnegative) from the `mathlib` in conjunction with lemma `pkk_nonneg` (from `general_lemmas.lean`).

```
theorem interval_integral.integral_nonneg : ∀ { μ : measure ℝ} {f : ℝ → ℝ}
{a b : ℝ},
  a ≤ b → ( ∀ (u : ℝ), u ∈ interval a b → 0 ≤ f u)
  → 0 ≤ ∫ (u : ℝ) in a..b, f u ∂μ
```

Code 4.35: from the `mathlib`

```
lemma pkk_nonneg (k: ℕ) {d: ℝ} (dge1: d ≥ 1):
    ∀ (x : ℝ), x ∈ set.interval 0 d → 0 ≤  (1 - d ⁻¹ * x) ^ k
```

Code 4.36: general_lemmas.lean

4.3.3 The Lemma `tr_seq_converges`

Finally, consider the lemma `tr_seq_converges`.

```
lemma tr_seq_converges
{ μ :measure ℝ}
( μ_spec_meas: is_spectral_measure RG A μ d)
[finite_measure μ]
(dge1: (d: ℝ) ≥ (1: ℝ))
: seq_limit ( λ k, @traced_algebra.tr ℝ RG _ _ tr_alg
                        (polynomial.eval₂ algebra.to_ring_hom A (pk k)))
          ( μ {0}).to_real
```

Code 4.37: general_RC.lean

Its proof is quite long and often technical. We will thus only point out the main steps. After rewriting using μ`spec_meas` and simplifying, we obtain the goal

```
seq_limit ( λ (k : ℕ), ∫ (x : ℝ) in Icc 0 ↑d, (1 - ↑(d)⁻¹ * x) ^ k ∂μ)
 ( ⇑μ {0}).to_real
```

This claim can then be solved by the dominated convergence theorem. We thus define the following.

```
let F : ℕ → ℝ → ℝ
      := ( λ k, λ x, polynomial.eval x (pk k)),
let f : ℝ → ℝ
      := ( λ x, ite ((x: ℝ)=0) 1 0),
let bound : ℝ → ℝ
      := ( λ x, 1),
```

```
let μ':= μ.restrict (Icc 0 (d: ℝ)),
```

Code 4.38: general_RC.lean

The proof then establishes $(F_n)_{n\in\mathbb{N}}$ and f are (everywhere, thus almost everywhere) measurable functions, and that `bound` is integrable. Moreover, `bound` is an upper bound of the sequence $(F_n)_{n\in\mathbb{N}}$ and this sequence converges (everywhere, thus almost everywhere) point-wise to f. Thus, the hypotheses of the bounded convergence theorem are satisfied and we can conclude.

```
have convergence :
   filter.tendsto ( λ (n : ℕ), ∫ (a : ℝ), ( λ (n : ℕ), F n) n a ∂μ') filter
   .at_top (nhds ( ∫ (a : ℝ), f a ∂μ'))
       := measure_theory.tendsto_integral_of_dominated_convergence
           bound F_ae_measurable f_ae_measurable bound_integrable
           h_bound h_lim,
```

Code 4.39: general_RC.lean

Note that the last three lines are the proof of the statement `convergence`, indicating that this follows from `measure_theory.tendsto_integral_of_dominated_convergence` using the facts established before.

The claim of the theorem then follows from the statement `convergence` by rewriting the integral of F_n and calculating the integral of f.

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Table of Notation

Selbständigkeitserklärung

Ich habe die Arbeit selbständig verfasst, keine anderen als die angegebenen Quellen und Hilfsmittel benutzt und bisher keiner anderen Prüfungsbehörde vorgelegt. Außerdem bestätige ich hiermit, dass die vorgelegten Druckexemplare und die vorgelegte elektronische Version der Arbeit identisch sind und dass ich von den in § 26 Abs. 6 vorgesehenen Rechtsfolgen Kenntnis habe.

Regensburg, 05.08.2021

Ort, Datum

(Matthias Uschold)