FACOLTÀ DI SCIENZE MATEMATICHE FISICHE E NATURALI
Corso di Laurea in Matematica

Rho Invariants and Classification Problems in Geometry

Relatore:
Prof.
PAOLO PIAZZA

Candidato:
ENRICO TOFFOLI
matricola 1286911

Sessione Autunnale
Anno Accademico 2013-2014
Dipartimento di Matematica 'Guido Castelnuovo'

Contents

Introduction

In 1963, Atiyah and Singer proved their celebrated index theorem, which expresses the analytical (Fredholm) index of a linear elliptic differential operator D on a closed manifold M in topological terms, depending only on M and on the homotopy class of the principal symbol of D. Their formula can be written as

$$\operatorname{ind} D = \int_M \mathrm{AS},$$

where AS is a differential form representing the topological data in question. The Atiyah-Singer index theorem represents an extraordinary bridge between analysis, geometry and topology. Moreover, it easily implies many important formulas such as those of Chern-Gauss-Bonnet, Hirzebruch-Riemann-Roch and the Hirzebruch signature formula.

The first proofs of the Atiyah-Singer index theorem made extensive use of either cobordism or topological K-theory, and were thus of global nature. Some years later, however, another approach arised, based on the heat equation. In fact, suppose that

$$D = \begin{pmatrix} 0 & D^- \\ D^+ & 0 \end{pmatrix}$$

is a so-called graded Dirac operator on M (important examples being the signature operator and the spin Dirac operator). Then, by a remark of McKean and Singer, the analytical index of its positive part D^+ can be expressed as a supertrace

$$\operatorname{ind} D^+ = \operatorname{str} e^{-tD^2} = \int_M \operatorname{str}_x \mathcal{E}(t;x,x)dx \qquad \text{for all } t > 0.$$

Here e^{-tD^2}, constructed by the spectral theorem for D, is the fundamental solution to the heat equation $\left(\frac{\partial}{\partial t} + D^2\right) f = 0$, and $\mathcal{E}$ is its (smooth) Schwartz kernel. McKean and Singer raised the following natural question: does the term $\operatorname{str}_x \mathcal{E}(t;x,x)dx$, thanks to some “fantastic cancellations”, converge to the differential form AS of Atiyah-Singer’s formula? The affirmative answer was given thanks to the work Patodi and Gilkey, and simplified in the following years by Atiyah, Bott, Patodi and Getzler among others. This new way of proving the index formula carries indeed new information, because the equality is shown at the *local* level. In fact, this last result is also known as the "local index theorem".

The heat equation methods carry over very well to many generalisations of the index theorem. For example, consider the case where M is an even dimensional manifold with boundary. In order to have a well defined index, then, it is necessary to impose boundary conditions on the differential operator. While an index theorem for classical local boundary conditions exists and can be proven by topological methods, it requires the vanishing of a certain obstruction. This obruction is nonzero for the usual elliptic operators of Riemannian geometry. In particular, for example, local boundary conditions fail to give a generalisation for the Hirzebruch signature theorem to manifolds with boundary. To circumvent this problem, in 1974 Atiyah, Patodi and Singer, using heat kernel methods, solved the index problem for manifolds with boundary under some new *global* boundary conditions, which involve a spectral projection associated to the boundary operator. Under these boundary conditions, the index of the positive part of a Dirac operator is then computed as

$$\operatorname{ind} D^+ = \int_M \mathrm{AS} - \frac{h(B) + \eta(B)}{2},$$

where $h(B)$ is the dimension of the null-space of the boundary operator B. The other correction term $\eta(B)$, called the η-invariant, was introduced by the above authors and can ve viewed as a measure of the "spectral asymmetry" of B between positive and negative eigenvalues. With some effort, then, the Atiyah-Patodi-Singer formula for the signature operator ($d + \delta$ with an appropriate grading) applies to give a signature theorem for manifolds with boundary, involving the eta invariant of the so-called odd signature operator on ∂M.

Another possible extension of index theory is for covering spaces $\widetilde{M} \to M$ of closed manifolds. If the covering is finite of order m, the overlying manifold is closed as well, and Atiyah-Singer's classical theorem implies that the index of a lifted elliptic operator is m times the index of the underlying one. When the deck transformation group Γ is infinite, however, $\widetilde{M}$ is an open manifold and classical index theory does not apply: in general, a lifted elliptic operator on $\widetilde{M}$ is not even Fredholm anymore. In a seminal paper, Atiyah pointed out that we can use a Von Neumann trace, adapted to the Γ-action, to define a normalised index $\operatorname{ind}_\Gamma$, and recover an index theory on $\widetilde{M}$. In the same article, dated 1976, he proves his beautiful Γ-index formula

$$\operatorname{ind}_\Gamma \widetilde{D} = \operatorname{ind} D,$$

where $\widetilde{D}$ denotes the lift to $\widetilde{M}$ of the elliptic operator D on M. For Dirac operators, Atiyah's formula can be shown by heat equation methods, and this leads to the question of whether (even-dimensional) manifolds with boundary can successfully introduced in this context. This is indeed the case, but a solution to the corresponding Atiyah-Patodi-Singer problem was not given before 1993. In that year, Ramachandran generalised at once our last two formulas, showing (for a lift $\widetilde{D}$ of a Dirac operator D on M) that

$$\operatorname{ind}_\Gamma \widetilde{D}^+ = \int_M \mathrm{AS} - \frac{h_\Gamma(\widetilde{B}) + \eta_\Gamma(\widetilde{B})}{2}.$$

The correction terms here are suitable variations of those appearing in Atiyah-Patodi-Singer. As usual, one of the applications of this index theorem is towards a signature formula. On coverings of a compact manifold, the notion of a Γ-signature can then be defined and, thanks to the work of Luck and Schick (2002), Ramachandran's theorem for the signature operator can be used to obtain a Γ-signature theorem for manifolds with boundary.

The eta invariant $\eta(B)$ of the boundary operator, appearing in the index and signature formulas for even-dimensional manifolds with boundary, can in fact be defined for any Dirac operator on a closed odd dimensional Riemannian manifold N. A similar remark apply for a Γ-covering $\widetilde{N} \to N$ and the Γ-eta invariant $\eta(B)$ of a lifted Dirac operator. These invariants are not very stable. For example, when B is the odd signature operator on N, they both depend on the Riemannian metric. Taking the difference between them, however, we obtain the much more stable Γ-rho invariant

$$\rho_\Gamma(B) := \eta_\Gamma(\widetilde{B}) - \eta(B),$$

introduced by Cheeger and Gromov in 1985. For the signature operator, $\rho_\Gamma(B)$ appears to be an oriented diffeomorphism invariant. In particular, it can be used in geometric topology for manifold classification problems. Even if explicit computations are often difficult to achieve, Γ-rho invariants were succesfully employed to prove many interesting results, a typical example being the following theorem.

Theorem. (Chang, Weinberger; 2003) *Let M^{4k-1} a $(4k-1)$-dimensional closed smooth manifold ($k \geq 2$) such that $\pi_1(M)$ contains torsion. Then, there exists an infinite family $\{M_i\}_{i\in\mathbb{N}}$ of different smooth manifolds M_i which are hompotopy equivalent to M but not diffeomorphic to it.*

In this master's thesis, I tried to present the developments illustrated above, as well as some tangentially related topics. Chang-Weinberger's theorem constituted a big motivation for all of my work, even if I could only give an incomplete sketch of its proof (which is carried in the topological framework of surgery theory). At the beginning of each chapter, I give a more detailed description about its content. Abundant references can be found in the relevant sections. I wish to thank my advisor Paolo Piazza for suggesting this project and guiding me through its realisation.

Chapter 0

Dirac operators and the Atiyah-Singer index theorem

In this chapter we will review the definitions and the main results, most of which without proof, concerning Dirac operators on closed Riemannian manifolds. This will serve for fixing the notation and as a reference for the following chapters. In particular, we will present a brief description of the heat kernel approach to the classical Atiyah-Singer index theorem for a graded Dirac operator. Because of its centrality in the subsequent chapters, special attention will be given to the concrete case of the signature operator, for which the Atiyah-Singer theorem leads to a proof of the Hirzebruch signature theorem. The spin Dirac operator will also be defined, and its relation with metrics of positive scalar curvature on spin manifolds will be briefly presented.

The treatment is inspired by [21], where full proofs can be found.

0.1 Clifford bundles and Dirac operators

We will introduce here the geometrical framework of Dirac operators, and describe some of their main properties.

The analytical properties we will need are shared with a larger class of operators, that of *elliptic* operators. Elliptic theory on compact manifolds is usually carried by constructing a *parametrix* (that is, an inverse modulo smoothing operators) by the theory of *pseudodifferential operators*, and use this parametrix to gather the other results. However, for (generalised) Dirac operators, it is easy to prove directly the elliptic estimate (Proposition 0.3), from which most of the analytic theory can be derived. Since we are only concerned with Dirac operators, we will support this simplified approach.

0.1.1 Basic definitions and properties

Let V be a vector space equipped with a symmetric bilinear form q. We define the *Clifford algebra* $\mathrm{Cl}(V)$ of V as the solution to a universal problem. Namely, we take $\mathrm{Cl}(V)$ to be a unital algebra, equipped with a linear map $\varphi : V \to \mathrm{Cl}(V)$ satisfying $\varphi(v)^2 = -q(v,v)1$, which is universal in the following sense: if $\psi : V \to A$ is a linear map to another unital algebra A with the property $\varphi(v)^2 = -q(v,v)1$, then there is a unique algebra homomorphism $\mathrm{Cl}(V) \to A$ fitting into the commutative diagram

$$\begin{array}{ccc} V & \xrightarrow{\varphi} & \mathrm{Cl}(V) \\ & {}_{\psi}\searrow & \downarrow \\ & & A \end{array}$$

Uniqueness modulo isomorphism is then implied by the universal property, while an explicit construction for $\mathrm{Cl}(V)$ is obtained by quotienting the tensor algebra $\mathcal{T}(V)$ by the relations $v \otimes v = -q(v,v)1$ for every $v \in V \hookrightarrow \mathcal{T}(V)$.

We will be interested in the case where V is a real vector space and q is a scalar product. In this case, we define a *Clifford module* S to be a left module over the complex algebra $\mathrm{Cl}(V) \otimes_{\mathbb{R}} \mathbb{C}$. equivalently, S is a complex vector space equipped with an $\mathbb{R}$-linear map $c : V \to End_{\mathbb{C}}(S)$ such that $c(v)^2 = -(v,v)1$ for any $v \in V$.

Let now M be a Riemannian manifold, possibly with boundary. Then TM is Riemannian vector bundle and we can naturally form a smooth bundle of (Clifford) algebras $\mathrm{Cl}(TM)$, with fiber at $x \in M$ given by $\mathrm{Cl}(T_xM)$. A bundle of Clifford modules S over $\mathrm{Cl}(TM)$ (that is, a smooth bundle of modules over $\mathrm{Cl}(TM) \otimes \mathbb{C}$) is called a *Clifford bundle* on M if:

1. S is equipped with a Hermitian metric $(\cdot,\cdot)$ such that the Clifford action of tangent vectors is skew-adjoint; that is, for any $v \in T_xM$ and $s_1, s_2 \in S_x$ we have
$$(vs_1, s_2) = -(s_1, vs_2);$$
2. S has a connection ∇ which is compatible with the metric and also respects the following compatibility condition with the Levi-Civita connection on M (also denoted by ∇): for any X, Y vector fields and section f of S, we have
$$\nabla_X(Yf) = (\nabla_X Y)f + Y\nabla_X f. \tag{0.1}$$

To any Clifford bundle S on M it is naturally associated a first order differential operator $D : C^\infty(M,S) \to C^\infty(M,S)$, called the *Dirac operator*. It is defined by the composition

$$C^\infty(M,S) \xrightarrow{\nabla} C^\infty(M, S \otimes T^*M) \to C^\infty(M, S \otimes TM) \to C^\infty(M,S),$$

where the firse arrow is given by the connection on S, the second by the Riemannian identification of tangent and contangent bundle, and the third is the

Clifford action of the tangent bundle on S. If $\{e_i\}$ is any local orthonormal base of vector fields, the Dirac operator D is immediately seen to be given by

$$Df = \sum_i e_i \nabla_i f \tag{0.2}$$

(where ∇_i denotes the covariant derivative ∇_{e_i}). The first important property of Dirac operators is the following, which is easily checked using a local orthonormal frame field and integration by parts.

Proposition 0.1. *Any Dirac operator D is formally self-adjoint. That is,*

$$\int_M (Df(x), g(x))dx = \int_M (f(x), Dg(x))dx$$

for any smooth sections f, g with compact support contined in the interior of M.

The main result which ties the Dirac operator with the geometry of the bundle is the so called *Weitzenböck formula*.

Proposition 0.2. *Let D a Dirac operator. Then*

$$D^2 = \nabla^* \nabla + K,$$

where $\nabla^* : C^\infty(M, S \otimes T^*M) \to C^\infty(M, S)$ is the formal adjoint of the connection and $K : S \to S$ is an endomorphism.

This formula descends directly from the definition of D. K can be easily expressed in terms of the curvature of ∇ and of the Clifford action.

0.1.2 Analysis of the Dirac operator

We shall now consider a slightly broader class of operators. Let S any Hermitian bundle on M, with compatible connection ∇. A first-order differential operator $D : C^\infty(M, S) \to C^\infty(M, S)$ is called a *generalised Dirac operator* if

$$D^2 = \nabla^* \nabla + K,$$

where K is now either an endomorphism or a first-order differential operator.

Note that, in this context, D needs not be formally self-adjoint. However, its formal adjoint D^* is also a generalised Dirac operator. Thanks to Weitzenböck formula (Proposition 0.2), any Dirac operator is a (formally self-adjoint) generalised Dirac operator.

We will now describe some properties of generalised Dirac operators on *closed* Riemannian manifolds (that is, compact without boundary). A first fundamental result is the following, concerning the Sobolev norms (for a definition of Sobolev spaces on compact manifolds, see [21]).

Proposition 0.3. (Elliptic estimate) *Let D a generalised Dirac operator over a closed manifold M. Then, for any $s \in \mathbb{N}$ there exist $C_s > 0$ such that*

$$\|f\|_{W^{s+1}} \leq C_s(\|f\|_{W^s} + \|Df\|_{W^s})$$

for all $f \in C^\infty(M, S)$.

Let $\mathcal{D}$ denote the Hilbert closure of D, viewed as an unbounded operator on $L^2(M, S)$, and let $\mathcal{D}^*$ be the closure of its formal adjoint D^*. Note that, thanks to the elliptic estimate for $s = 0$, also known as *Gårding inequality*, both their domains coincide with $W^1(M, S)$. Another consequence of Gårding inequality is the following proposition, which is usually proved by approximation with smooth functions.

Proposition 0.4. *Let D a generalised Dirac operator on a closed manifold M and $\mathcal{D}$, $\mathcal{D}^*$ as above. Then, $\mathcal{D}$ and $\mathcal{D}^*$ are mutually adjoint (as unbounded operators).*

In particular, the closure of any formally self-adjoint generalised Dirac operator is self-adjoint. This is of course the case for true Dirac operators associated to Clifford bundles.

Using the elliptic estimate, thanks to the Sobolev embedding theorem we have the following fundamental result.

Proposition 0.5. *Let D be a generalised Dirac operator on a compact manifold M, and let $\mathcal{D}$ denote its closure. Then,*

$$\ker \mathcal{D} \subseteq C^\infty(M, S).$$

Combining all the properties presented so far, we obtain the main theorem of this section.

Theorem 0.6. *Let D be a formally self-adjoint generalised Dirac operator on a closed manifold M. Then, the eigenvalues λ of $\mathcal{D}$ form a discrete subset of $\mathbb{R}$. Each eigenspace $H_\lambda \subseteq L^2(M, S)$ is finite dimensional and made of smooth sections. Moreover, there is an orthogonal decomposition*

$$L^2(M, S) = \bigoplus_\lambda H_\lambda. \tag{0.3}$$

Proof. (*Sketch*) Let H denote the Hilbert space $L^2(M, S)$, and $G \subseteq H \oplus H$ the graph of $\mathcal{D}$. Define Q as the composition:

$$H \xrightarrow{i_1} H \oplus H \xrightarrow{\pi_G} H \oplus H \xrightarrow{\pi_1} H,$$

where i_1 is the injection $f \to (f, 0)$, π_G is the orthogonal projection onto G and π_1 is the projection onto the first summand. Since $\|\pi_G\| = 1$, for any $f \in H$ we have

$$\|Qf\|^2 + \|\mathcal{D}Qf\|^2 = \|(Qf, \mathcal{D}Qf)\|^2 \leq \|(f, 0)\|^2 = \|f\|^2.$$

Hence, by Gårding's inequality (Proposition 0.3), Q is bounded as a map from L^2 to W^1. By Rellich's theorem, thus, it is compact as an operator on $H = L^2(M, S)$. Moreover, it is easily seen to be positive, self-adjoint, injective and of norm 1. By the spectral theorem for compact self-adjoint operators, Q has real eigenvalues $0 < \mu \leq 1$ accumulating only at 0, with finite dimensional eigenspaces H'_μ such that $H = \bigoplus_\mu H'_\mu$.

Let now $f \in H'_\mu$ for some eigenvalue μ. Using the fact thet $\mathcal{D}$ is self-adjoint (Proposition 0.4), it is not hard to find, from the definition of Q, that $\mathcal{D}f \in \operatorname{Dom}\mathcal{D}$ and $\mathcal{D}^2 f = (1-\mu)/\mu f$. Hence, the sections $\sqrt{1-\mu} f \pm \sqrt{\mu}\mathcal{D}f$ are eigensections of $\mathcal{D}$ of eigenvalues $\pm\lambda := \pm\sqrt{1-\mu}/\sqrt{\mu}$. Decomposition (0.3) into finite dimensional eginspaces follows, and the discreteness of the eigenvalues is due to the μ's accumulating only at 0.

To prove that each eigenspace is made of smooth sections, notice that $H_\lambda = \ker(\mathcal{D} - \lambda)$. Since $\mathcal{D} - \lambda$ is also a generalised Dirac operator, the thesis follows from Proposition 0.5. □

Corollary 0.7. *Let $D : C^\infty(M, S) \to C^\infty(M, S)$ be a formally self-adjoint generalised Dirac operator on a closed manifold M. Then, there exists an operator*

$$R : C^\infty(M, S) \to C^\infty(M, S)$$

(called a parametrix *for D) such that:*

(a) the operators $DR - 1$ and $RD - 1$ have smooth integral kernels;

(b) R extends to a bounded operator $R : W^s \to W^{s+1}$ for any s.

Proof. (*Sketch*) Let $L^2(M, S) = \bigoplus_\lambda H_\lambda$ as in (0.3). Define $R : H_\lambda \to H_\lambda$ to be equal to $1/\lambda$ for $\lambda \neq 0$ and 0 for $\lambda = 0$. Property (b) follows from the elliptic estimate, and hence R sends smooth sections into smooth sections thanks to the Soboloev embedding theorem.

To establish (a), notice that $DR - 1 = RD - 1 =$ projection on $\ker\mathcal{D}$. Since $\ker\mathcal{D}$ is finite dimensional and made of smooth sections, orthogonal projection to it has a smooth integral kernel. □

The parametrix constructed in Corollary 0.7 is also known as the *Green operator* for D. It is not the unique operator with the properties listed above. In fact, at some point we will need the existence of a parametrix whose kernel has a better behaviour.

0.1.3 Gradings and indices

In order to study interesting index problems for Dirac operators, we need one more piece of structure. A Clifford module S on a manifold M is called *graded* if it can be written as a direct sum

$$S = S^+ \oplus S^-$$

such that:

- S^+ and S^- are orthogonal with respect to the metric on S;
- the connection ∇ on S respects the decomposition; that is,

$$\nabla : C^\infty(M, S^\pm) \to C^\infty(M, S^\pm \otimes T^*M);$$

- the Clifford action of tangent vectors exchanges S^+ with S^-.

A grading on S can be obtained as follows. Let $\varepsilon : S \to S$ a bundle involution (that is: $\varepsilon^2 = 1$) such that

- ∘ ε is self-adjoint with respect to the metric on S;
- ∘ ε is *parallel* with respect to the connection; that is, $\nabla_X \varepsilon = \varepsilon \nabla_X$ for each vector bundle X;
- ∘ ε anticommutes with the Clifford action of tangent vectors.

Such an ε is called a *grading operator*. Then, the eigenbundles $S^\pm$ relative to the eigenvalues ± 1 of ε constitute a grading for S. Conversely, from any grading we can define a grading operator by taking $\varepsilon = 1$ on S^+ and $\varepsilon = -1$ on S^-.

The Dirac operator associated to a graded Clifford bundle decomposes in the following manner. Let $D^+ := D|_{S^+}$ and $D^- := D|_{S^-}$. Then D^+ is given by the composition $C^\infty(M, S^+) \xrightarrow{\nabla} C^\infty(M, S^+ \otimes T^*M) \to C^\infty(M, S^+ \otimes TM) \to C^\infty(M, S^-)$, and hence

$$D^+ : C^\infty(M, S^+) \to C^\infty(M, S^-).$$

Similarly,

$$D^- : C^\infty(M, S^-) \to C^\infty(M, S^+).$$

Since D is formally self-adjoint, D^+ and D^- are formally adjoints one of each other. Calling $\mathcal{D}^+$ and $\mathcal{D}^-$ the Hilbert closures closures of these operators (as unbounded operators between $L^2(M, S^+)$ and $L^2(M, S^-)$, from Proposition 0.4 we obtain:

Proposition 0.8. *$\mathcal{D}^+$ and $\mathcal{D}^-$ are mutually adjoint.*

Of course, opportune restrictions of the parametrix R of Corollary 0.7 give parametrices R^+ and R^- for D^+ and D^- respectively.

Proposition 0.9. *Let $D : C^\infty(M, S) \to C^\infty(M, S)$ a graded Dirac operator on a closed manifold M. Then, there exist operators*

$$R^\pm : C^\infty(M, S^\mp) \to C^\infty(M, S^\pm)$$

such that:

(a) the operators $D^\pm R^\pm - 1$ and $R^\pm D^\pm - 1$ have smooth integral kernels;

(b) $R^\pm$ extends to a bounded operator $R^\pm : W^s \to W^{s+1}$ for any s.

Regard now $\mathcal{D}^+ : W^1(M, S^+) \to L^2(M, S^-)$ and $\mathcal{D}^- : W^1(M, S^-) \to L^2(M, S^+)$ as bounded operators between Hilbert spaces. Then:

Proposition 0.10. *$\mathcal{D}^+$ and $\mathcal{D}^-$ are Fredholm operators, and the index of $\mathcal{D}^+$ is computed as*

$$\operatorname{ind} \mathcal{D}^+ = \dim\ker \mathcal{D}^+ - \dim\ker \mathcal{D}^- = \dim\ker D^+ - \dim\ker D^-.$$

Proof. Proposition 0.9 gives inverses modulo smoothing (and hence compact) operators for $\mathcal{D}^+$ and $\mathcal{D}^-$. By Atkinson theorem, then, $\mathcal{D}^+$ and $\mathcal{D}^-$ are Fredholm operators. Since they are mutually adjoint as unbounded operators, $\ker \mathcal{D}^- = (\operatorname{im} \mathcal{D}^+)^\perp$. Moreover, the null-spaces of $\mathcal{D}^+$ and $\mathcal{D}^-$ coincide with those of D^+ and D^- respectively, thanks to Proposition 0.5. Hence

$$\begin{aligned}\operatorname{ind} \mathcal{D}^+ &= \dim\ker \mathcal{D}^+ - \dim\operatorname{coker} \mathcal{D}^+ = \dim\ker \mathcal{D}^+ - \dim\ker \mathcal{D}^- = \\ &= \dim\ker D^+ - \dim\ker D^-.\end{aligned}$$

□

We define

$$\operatorname{ind} D^* := \operatorname{ind} \mathcal{D}^+$$

and call this integer the *index* of D^+.

Remark 0.11. *Thanks to Proposition 0.7, the closure $\mathcal{D} : W^1(M, S) \to L^2(M, S)$ of the full Dirac operator D is also a Fredholm operator. However, being a self-adjoint operator, its index is always 0 and this does not drive us to an interesting index theory.*

0.1.4 Example: the signature operator

We will now study an important example of Dirac operator, and see how its index coincide with a classical invariant in topology.

Let M^n an oriented Riemannian manifold (for the moment we do not suppose M to be compact or without boundary). Let $\Lambda^* := \Lambda^*(T^*M) \otimes \mathbb{C}$ the complixified exterior bundle and *vol* be the Riemannian volume form of M. Λ^* has then a Hermitian metric coming naturally from the Riemannian metric on M. We recall that the *Hodge star operator* is the only bundle isomorphism $\star : \Lambda^* \to \Lambda^*$ such that $\star(\Lambda^p) \subseteq \Lambda^{n-p}$ and

$$\alpha \wedge \star\overline{\beta} = (\alpha, \beta)\,\mathrm{vol} \qquad \forall \alpha,\, \beta \in \Lambda^p_x.$$

$\star$ is almost an involution, in the sense that

$$\star^2 = (-1)^{np+p} \qquad \text{on } \Lambda^p.$$

Λ^* is in fact a Clifford bundle (with connection inherited by the Levi-Civita connection) if we consider the action of $v \in T_xM$ on $\omega \in \Lambda^*_x$ given by

$$c(v)\omega := \nu \wedge \omega + \nu \lrcorner\, \omega = \nu \wedge \omega - \star(\nu \wedge \star\omega),$$

where $\nu \in T_x^*M$ is the dual of $v \in T_xM$ through the isomorphism given by the metric.

By choosing a local orthonormal frame of vector fields $\{v_i\}$, with dual basis $\{\nu_i\}$, using (0.2) we can easily see that the Dirac operator of Λ^* is given by

$$D\omega = \sum_i c(v_i)\nabla_i\omega = \sum_i \nu_i \wedge \nabla_i\omega + \sum_i \nu_i \lrcorner \nabla_i\omega = d\omega + \delta\omega,$$

where $\delta = (-1)^{np+n+p} \star d\star$ (on p-forms) is the formal adjoint of the exterior differential d. Hence $D = d + \delta$ and $D^2 = d\delta + \delta d =: \Delta$ is the Hodge Laplacian. D acts on the space $\Omega^*(M) := C^\infty(M, \Lambda^*)$ of complex differential forms on M.

If $M = M^{2l}$ is even-dimensional, we can introduce a grading on Λ^* as follows. Let $\tau : \Lambda^* \to \Lambda^*$ be defined by

$$\tau := i^{p(p-1)+l} \star \qquad\qquad \text{on } \Lambda^p.$$

τ is then an involution and has all the properties listed above for grading operators. Λ^* splits hence as

$$\Lambda^* = \Lambda^+ \oplus \Lambda^-,$$

where Λ^+ and Λ^- are respectively the $+1$ and -1 eigenbundles of τ. Correspondingly, the space of complex differential forms splits as

$$\Omega^* = \Omega^+ \oplus \Omega^-,$$

and the Dirac operator is written as

$$D = \begin{pmatrix} 0 & D^- \\ D^+ & 0 \end{pmatrix}.$$

When considered with this particular grading, the Dirac operator is called the (even) *signature operator* on M. We will see the reason in a moment. The Hodge Laplacian also splits as

$$\Delta = \begin{pmatrix} D^-D^+ & 0 \\ 0 & D^+D^- \end{pmatrix} = \begin{pmatrix} \Delta^+ & 0 \\ 0 & \Delta^- \end{pmatrix}.$$

Suppose from now on that M is a closed manifold. Formal self-adjointness of D then yelds $\ker D = \ker \Delta$. This is the space of *harmonic forms* and is usually denoted as $\mathcal{H}^*(M)$. It also decomposes as

$$\mathcal{H}^*(M) = \mathcal{H}^+(M) \oplus \mathcal{H}^-(M),$$

where $\mathcal{H}^\pm(M) := \ker \Delta^\pm = \ker D^\pm$. By our definition (1.34), we have then

$$\operatorname{ind} D^+ = \dim \mathcal{H}^+(M) - \dim \mathcal{H}^-(M). \tag{0.4}$$

Recall the following version of Hodge theorem.

Theorem 0.12. (Hodge) *Let M a closed oriented Riemannian manifold, and let $H^*(M)$ denote the complex de Rham cohomology of M. Then, the map*

$$\mathcal{H}^*(M) \to H^*(M)$$
$$\omega \to [\omega]$$

is an isomorphism of graded complex vector spaces.

A surprisingly easy proof of this classical theorem, based on the analytical results of the previous section, can be found in [21]. Suppose now that the dimension of M is a multiple of 4 ($M = M^{2l} = M^{4k}$). Being a closed oriented manifold, we can non-trivially define its *signature.* To to do this in the context of de Rham cohomology with complex coefficients, we can consider the nondegenerate Hermitian form

$$h : H^{2k}(M) \times H^{2k}(M) \to \mathbb{C}$$
$$([\alpha], [\beta]) \mapsto \int_M \alpha \wedge \overline{\beta},$$

called the *intersection form* on M. The signature of M is then defined as the number

$$\operatorname{sign} M := \operatorname{sign}(h) = p - q,$$

where p is the number of $+1$'s and q is the number of -1's in the canonical diagonalised form of h. By the Hodge theorem, to define the signature we can equivalently use the corresponding Hermitian form on

$$\mathcal{H}^{2k}(M) \times \mathcal{H}^{2k}(M) \to \mathbb{C}.$$

This identification will be the main ingredient in the proof of the next proposition. Notice that, thanks to de Rham's theorem, the signature is an oriented topological invariant of M.

Proposition 0.13. *Let M^{4k} a closed oriented Riemannian manifold of dimension $4k$, and let $D = d + \delta$ the signature operator on M. Then,*

$$\operatorname{ind} D^+ = \operatorname{sign} M.$$

Proof. For $0 \leq p \leq 2k$, let

$$\Lambda_p := (\Lambda^p + \Lambda^{4k-p}),$$

so that $\Lambda^* = \bigoplus_{p=1}^{2k} \Lambda_p$, and let Λ_p^+ and Λ_p^- be its graded parts. Let $\mathcal{H}_p(M)$, $\mathcal{H}_p^+(M)$ and $\mathcal{H}_p^-(M)$ be defined accordingly.

By definition of our grading, we can write a section α of $\Lambda_p^\pm$ as $\alpha = \beta \pm \tau\beta$, with β a p-form (τ is the grading operator). Since Δ preserves the p-grade and commutes with τ, for $p < 2k$ α is harmonic if and only if $\Delta\beta = 0$. We have thus isomorphisms

$$\mathcal{H}_p^+(M) \cong \mathcal{H}^p(M) \cong \mathcal{H}_p^-(M) \qquad \text{for } p < 2k.$$

In particular,

$$\operatorname{ind} D^+ = \dim \mathcal{H}^+(M) - \dim \mathcal{H}^-(M) = \sum_{p=1}^{2k} \dim \mathcal{H}_p^+(M) - \sum_{p=1}^{2k} \dim \mathcal{H}_p^-(M) =$$
$$= \dim \mathcal{H}_{2k}^+(M) - \dim \mathcal{H}_{2k}^-(M). \tag{0.5}$$

Notice that, on Λ^{2k}, the involution $\tau = i^{p(p-1)+l}\star$ coincides with $\star$. Hence, by definition of the grading, $\star = +1$ on Λ_{2k}^+ and $\star = -1$ on Λ_{2k}^-. Let h the intersection pairing on $\mathcal{H}^{2k}(M) \times \mathcal{H}^{2k}(M)$, for which, thanks to the Hodge theorem, we have $\operatorname{sign}(h) = \operatorname{sign} M$. Then,

$$h(\alpha, \alpha) = \int_M \star\alpha \wedge \overline{\alpha} = \|\alpha\|_{L^2}^2 > 0 \qquad \text{for } \alpha \in \mathcal{H}_{2k}^+(M)\backslash\{0\}$$

and

$$h(\alpha, \alpha) = \int_M - \star \alpha \wedge \overline{\alpha} = -\|\alpha\|_{L^2}^2 < 0 \qquad \text{for } \alpha \in \mathcal{H}_{2k}^-(M)\backslash\{0\}.$$

It follows that the signature of M is given by

$$\operatorname{sign} M = \operatorname{sign}(h) = \dim \mathcal{H}_{2k}^+(M) - \dim \mathcal{H}_{2k}^-(M). \tag{0.6}$$

Combining (0.5) and (0.6), we get

$$\operatorname{ind} D^+ = \dim \mathcal{H}_{2k}^+(X) - \dim \mathcal{H}_{2k}^+(X) = \operatorname{sign} X.$$

□

As a consequence of Proposition 0.13, we notice that the index of the signature operator does not depend on the Riemannian metrics used to define it.

0.1.5 Example: spin groups and the spin Dirac operator

On some manifolds, called spin manifolds, another structure of Clifford bundle arises in a natural way. We will briefly give the necessary definitions and then study a relation between the index of this Dirac operator and the problem of existence of metrics of positive scalar curvature on manifolds.

Consider the Clifford algebra $\mathrm{Cl}(n)$ associated to the vector space $\mathbb{R}^n$ and its canonical inner product, which is linearly generated by the products $e_{j_1} \cdots e_{j_k}$ (e_1,...,e_n being the canonical base of $\mathbb{R}^n$). It is a *superalgebra*, in the sense that we can write $\mathrm{Cl}(n) = \mathrm{Cl}_0(n) \oplus \mathrm{Cl}_1(n)$, with $\mathrm{Cl}_i(n)\,\mathrm{Cl}_j(n) \subseteq \mathrm{Cl}_{i+j}(n)$ (we think the indices as elements of $\mathbb{Z}/2\mathbb{Z}$). To do so, it suffices to define $\mathrm{Cl}_0(n) := \mathrm{Span}\{e_{j_1} \cdots e_{j_k} \mid k \text{ is even}\}$, and $\mathrm{Cl}_1(n) := \mathrm{Span}\{e_{j_1} \cdots e_{j_k} \mid k \text{ is odd}\}$. From the formula $vv = -(v, v)$, we see that any $v \in \mathbb{R}^n$ is invertible in $\mathrm{Cl}(n)$. We define:

- the *pin group* $\mathrm{Pin}(n)$ as the multiplicative group generated in $\mathrm{Cl}(n)$ by the unit vectors of $\mathbb{R}^n$;
- the *spin group* $\mathrm{Spin}(n)$ as the index 2 subgroup of $\mathrm{Pin}(n)$ given by intersection with $\mathrm{Cl}_0(n)$.

We define the *grading homomorphism* ε of $\mathrm{Cl}(n)$ to be $+1$ on $\mathrm{Cl}_0(n)$ and -1 on $\mathrm{Cl}_1(n)$. There is a well defined homomorphism $\gamma : \mathrm{Pin}(k) \to O(k)$ given by $\gamma(a)v := av\varepsilon(a^{-1})$, which is easily seen to be surjective, and such that $\gamma(\mathrm{Spin}(n)) = SO(n)$. In fact, restriction to $\mathrm{Spin}(n)$ gives a double cover

$$0 \to \{\pm 1\} \to \mathrm{Spin}(n) \xrightarrow{\gamma} SO(n) \to 0. \tag{0.7}$$

We can now define a new class of manifolds in the following way. Let M^n be an oriented Riemannian manifold of dimension n, and let E the principal $SO(n)$-bundle of oriented orthonormal frames for its tangent bundle. A *spin structure* on M is a principal $\mathrm{Spin}(n)$-bundle $\widetilde{E}$ over M which is a double cover $\widetilde{E} \to E$ of E and such that the restriction to each fibre is isomorphic to the double cover $\mathrm{Spin}(n) \to SO(n)$ of (0.7). A manifold admitting a spin structure is called a *spin manifold*. There are topological obructions to the existence of spin structures on manifolds, that we will not discuss here, so that not every manifold is spin. When referring to a spin manifold, however, we will often assume that a particular spin structure is fixed.

There is a natural Clifford bundle associated to every even-dimensional spin manifold. It is defined by the associated bundle construction, through a suitable representation of the spin group. To define this piece of structure, we will use the following simple algebraic facts in the representation theory for complexified Clifford algebras.

Proposition 0.14. *If $n = 2k$ is even, the complexified Clifford algebra $\mathrm{Cl}(n)\times\mathbb{C}$ admits a unique irreducible complex representation Ψ, of dimension 2^k. Restriction of Ψ to $\mathrm{Spin}(n)$ gives a reducible group representation which can be written as the sum $\Psi = \Psi^+ \oplus \Psi^-$ of two irreducible representations of dimension 2^{k-1}.*

We call Φ the *spin representation*, and Ψ^+ and Ψ^- the *half spin representations*. Suppose now that M^n is an even dimensional spin manifold, with principal $\mathrm{Spin}(n)$-bundle $\widetilde{E}$. By means of the spin representation $\varphi : \mathrm{Spin}(n) \to GL(\Phi)$, we can define the *spinor bundle* $\not{S}(M)$ as

$$\not{S}(M) := \widetilde{E} \times_\varphi \Phi$$

(here $\times_\varphi$ denotes the associated bundle construction). When M is spin, the bundle of Clifford algebras $\mathrm{Cl}(TM)$ can be written as

$$\mathrm{Cl}(TM) = \widetilde{E} \times_{\mathrm{Ad}} \mathrm{Cl}(n),$$

where $\mathrm{Ad}_g(\xi) := gag^{-1}$ is the *adjoint representation*. By letting $\mathrm{Cl}(n)$ act on Φ by the spin representation, it follows that $\not{S}(M)$ is naturally a bundle of Clifford modules over $\mathrm{Cl}(TM)$. A compatible connection on $\not{S}(M)$ is defined from the principal connection lifted on $\widetilde{E}$ from the $SO(n)$ bundle of frames E. A metric, on which the Clifford action is skew-adjoint, is obtained because the group spin representation is unitary. These structures make indeed $\not{S}(M)$ a Clifford bundle. We will call the Dirac operator associated to $\not{S}(M)$ the *spin Dirac operator*, and denote it by $\not{D}$. The decomposition $\Phi = \Phi^+ \oplus \Phi^-$ gives $\not{S}(M)$ the structure of a graded Clifford bundle. As a consequence, the spin Dirac operator $\not{D}$ is a graded Dirac operator.

When applied to the spin Dirac operator, Weitzenböck formula (Proposition 0.2) becomes

$$\not{D}^2 = \nabla^*\nabla + \frac{1}{4}\kappa, \tag{0.8}$$

where κ is the *scalar curvature* of M, obtained by contracting twice the Riemannian curvature tensor of M. Since $\nabla^*\nabla$ is a positive operator, we see that, if κ is everywhere positive, $\not{D}$ must have trivial kernel. In particular, the index of $\not{D}^+$ must be 0 (this argument is due to Bochner). We will se how the Atiyah-Singer index theorem, combined with this remark, gives a topological obstruction to the existence of a metric of strictly positive scalar curvature on an even-dimensional spin manifold M.

0.2 The Atiyah-Singer index theorem

In this section we will briefly present the heat kernel approach to the Atiyah-Singer index theorem for Dirac operators on closed manifolds. As a corollary, we will deduce the classical Hirzebruch signature theorem. The main goal of Chapter 1 will be to generalise these two results to the case of manifolds with boundary.

0.2.1 The heat operator and McKean-Singer's formula

Let M^n be a closed Riemannian manifold, and let S a Clifford bundle on M, with associated Dirac operator $D : C^\infty(M,S) \to C^\infty(M,S)$.

We have seen in the previous section that the Hilbert closure $\mathcal{D}$ of D is a self-adjoint unbounded operator on $L^2(M,S)$. By the abstract theory of unbounded operators, $\mathcal{D}^2$ is positive and self-adjoint. By functional calculus, for any $t > 0$ we can hence define the bounded operator $e^{-t\mathcal{D}^2}$. Thanks to Theorem 0.6, $L^2(M,S)$ is a direct sum of eigenspaces H_λ of $\mathcal{D}$, of eigenvalue λ. If $f = \sum_\lambda f_\lambda$ with $f_\lambda \in H_\lambda$ for each λ, $e^{-t\mathcal{D}^2}$ acts on f as

$$e^{-t\mathcal{D}^2} f = \sum_\lambda e^{-t\lambda^2} f_\lambda. \tag{0.9}$$

It is not hard to see that such an operathor is *smoothing*. That is, it has a smooth integral kernel $\mathcal{E}(t;\cdot,\cdot) \in C^\infty(M \times M, S \boxtimes S^*)$, so that

$$(e^{-t\mathcal{D}^2} f)(x) = \int_M \mathcal{E}(t;x,y) f(y) dy.$$

$e^{-t\mathcal{D}^2}$ is also called the *heat operator* on M, and its kernel $\mathcal{E}$ is called the *heat kernel*. The reason for such names is that $e^{-t\mathcal{D}^2}$ constitutes a fundamental solution for the *heat equation*

$$\left(\frac{\partial}{\partial t} + D^2\right) f = 0 \tag{0.10}$$

on M. That is, $f(t,\cdot) := e^{-t\mathcal{D}^2} f_0$ is the unique solution of (0.10) with initial data $f(t,\cdot) = f_0$. Equivalently, $\mathcal{E}$ is the only kernel satisfying (0.10) on the first space variable and such that

$$\int_M \mathcal{E}(t;x,y) f(y) dy \xrightarrow{t\to 0} f(y)$$

uniformly on y. Since $e^{-t\mathcal{D}^2}$ is smoothing for any $t > 0$, it is in particular a *trace class* operator. Thanks to (0.9), we have

$$\operatorname{tr} e^{-t\mathcal{D}^2} = \sum_\lambda e^{-t\lambda^2}.$$

In terms of the heat kernel $\mathcal{E}$, the trace appears to be given by

$$\operatorname{tr} e^{-t\mathcal{D}^2} = \int_M \operatorname{tr}_x \mathcal{E}(t;x,x) dx, \tag{0.11}$$

where tr_x denotes the fibre trace in $S_x \otimes S_x^*$.

Suppose now that $S = S^+ \oplus S^-$ is graded, with grading operator τ. We can hence define the *graded heat operators* $e^{-t\mathcal{D}^-\mathcal{D}^+}$ and $e^{-t\mathcal{D}^+\mathcal{D}^-}$, which satisfy similar properties to those of $e^{-t\mathcal{D}^2}$ and such that

$$e^{-t\mathcal{D}^+\mathcal{D}^-} = e^{-t\mathcal{D}^-\mathcal{D}^+} \oplus e^{-t\mathcal{D}^+\mathcal{D}^-}$$

holds. We can also define the *supertrace* of a trace class operator A on $L^2(M,S)$ to be

$$\operatorname{str} A = \operatorname{tr}(\tau A).$$

The following equalities easily follow from the definition:

$$\operatorname{str} e^{-t\mathcal{D}^2} = \operatorname{tr} e^{-t\mathcal{D}^-\mathcal{D}^+} - \operatorname{tr} e^{-t\mathcal{D}^+\mathcal{D}^-} = \int_M \operatorname{str}_x \mathcal{E}(t;x,x) dx \tag{0.12}$$

(where $\operatorname{str}_x(\cdot) = \operatorname{tr}_x(\tau\,\cdot)$). The next result is crucial in our treatment, as it relates the heat operator with the index problem.

Proposition 0.15. (McKean-Singer formula) *Let D a graded Dirac operator on a closed Riemannian manifold M. Then, for any $t > 0$, we have*

$$\operatorname{ind} D^+ = \operatorname{str} e^{-t\mathcal{D}^2}. \tag{0.13}$$

Proof. By (0.12) we have

$$\operatorname{str} e^{-t\mathcal{D}^2} = \sum_{\lambda} e^{-t\lambda^2}(n^+(\lambda) - n^-(\lambda)) = \operatorname{ind} D^+ + \sum_{\lambda \neq 0} e^{-t\lambda^2}(n^+(\lambda) - n^-(\lambda)),$$

where $n^{\pm}(\lambda)$ denotes the multiplicity of λ^2 as an eigenvalue of $\mathcal{D}^{\mp}\mathcal{D}^{\pm}$. If we show that the corresponding eigenspaces E_λ^+ and E_λ^- are isomorphic for $\lambda \neq 0$, a big cancellation will occur and (0.13) will be proven.

Notice hence that $\mathcal{D}^+$ sends E_λ^+ to E_λ^-. In fact, if $f \in E_\lambda^+$, we have

$$\mathcal{D}^+\mathcal{D}^-(\mathcal{D}^+ f) = \mathcal{D}^+(\mathcal{D}^-\mathcal{D}^+ f) = \lambda^2 \mathcal{D}^+ f.$$

If $\lambda \neq 0$, $\lambda^{-2}\mathcal{D}^- : E_\lambda^- \to E_\lambda^+$ clearly is an inverse to $\mathcal{D}^+$, giving us the desired isomorphism $E_\lambda^+ \cong E_\lambda^-$. □

Thanks to (0.12), McKean-Singer's formula can also be read as follows: for any $t > 0$, we have

$$\operatorname{ind} D^+ = \int_M \operatorname{str}_x \mathcal{E}(t; x, x) dx.$$

Since the integral on the right does not depend on t, we can clearly write

$$\operatorname{ind} D^+ = \lim_{t \to 0} \int_M \operatorname{str}_x \mathcal{E}(t; x, x) dx. \tag{0.14}$$

The next step is to study the asymptotics of the heat kernel $\mathcal{E}$ as $t \to 0$.

0.2.2 The asymptotics expansion for the heat kernel

Let $\phi : \mathbb{R}^+ \to X$ a function with values in a Banach space X. A formal series $\sum_k a_k$ (with $a_k : \mathbb{R}^+ \to X$ for any $k \geq 0$) is called an *asymptotic expansion* for f as $t \to 0$ if, for each positive n, there exist j_n such that, for sufficiently small t,

$$j \geq j_n \implies \left\| \phi(t) - \sum_{k=0}^{l} a_k(t) \right\| \leq C_{j,n} t^n$$

for some constant $C_{j,n}$. In such case, we write

$$\phi(t) \sim \sum_{k=0}^{\infty} a_k(t).$$

The next fundamental result can be proven by purely geometric means (see [21]).

Proposition 0.16. *Let M^n a closed Riemannian manifold of dimension n, and let D a graded Dirac operator on M with heat kernel $\mathcal{E}$. Then, $\mathcal{E}$ has an asymptotic expansion of the form*

$$\mathcal{E}(t; x, y) \sim h(t; x, y) \sum_{k \geq 0} \Theta_k(x, y) t^k, \tag{0.15}$$

where

- $h(t;x,y) := \frac{1}{(4\pi t)^{n/2}} e^{-d(x,y)^2/4t}$;
- Θ_k *is a smooth time-depending section of* $S \boxtimes S^*$ *for each* k;
- *the values* $\Theta(x,x)$ *along the diagonal can be written as algebraic expressions of the metrics and connection coefficients.*

The asymptotic expansion is valid in the topology of $C^r(M \times M, S \boxtimes S^*)$ *for each* $r \geq 0$.

Remark 0.17. *It can be shown that* $\Theta_0(x,x) \equiv 1$. *This is not surprising, as the heat kernel on* $\mathbb{R}^n$ *is given exactly by* $h(t;x,y) = \frac{1}{(4\pi t)^{n/2}} e^{|x-y|^2/4t}$.

A first consequence of the asymptotic expansion is the following, which we will use systematically in Chapter 1.

Corollary 0.18. *Let* $\{\varphi_\lambda\}$ *be an orthonormal base of* $L^2(M,S)$ *made of eigensections of* D *of eigenvalues* λ *(which exists by Theorem 0.6). Then, there exists* $C > 0$ *such that, for any* $x \in M$ *and small enough* t,

$$\sum_\lambda e^{-t\lambda^2} |\varphi_\lambda(x)|^2 = \operatorname{tr}_x \mathcal{E}(t;x,x) < Ct^{-n/2}. \tag{0.16}$$

Moreover, there exists $C' > 0$ *such that*

$$\sum_\lambda e^{-t\lambda^2} = \operatorname{tr} e^{-tD^2} < C' t^{-n/2}. \tag{0.17}$$

Proof. It is clear from (0.9) that the heat kernel can be written as

$$\mathcal{E}(t;x,y) = \sum_\lambda e^{-t\lambda^2} \varphi_\lambda(x) \otimes \varphi_\lambda(y)^*,$$

where * denotes the duality induced by the metrics. In particular,

$$\sum_\lambda e^{-t\lambda^2} |\varphi_\lambda(x)|^2 = \sum_\lambda e^{-t\lambda^2} \varphi_\lambda(x) \otimes \varphi_\lambda(x)^* = \operatorname{tr}_x \mathcal{E}(t;x,x).$$

From Proposition 0.16, we get an asymptotic expansion

$$\operatorname{tr}_x \mathcal{E}(t;x,x) \sim \frac{1}{(4\pi t)^{-n/2}} \sum_{k \geq 0} \operatorname{tr}_x \Theta_k(x,x) t^k,$$

which is valid in the Banach space $C^r(M)$ for any r. (0.16) easily follows.

For what concerns (0.17), we already know that the equality holds. Thanks to (0.11), moreover, we get an asymptotic expansion

$$\operatorname{tr} e^{-tD^2} \sim \frac{1}{(4\pi t)^{-n/2}} \sum_{k \geq 0} t^k \int_M \operatorname{tr}_x \Theta_k(x,x) dx, \tag{0.18}$$

as a function with values in $\mathbb{R}$, from which the desired inequality follows. □

Of immediate interest for this chapter is the following other important consequence.

Corollary 0.19. *Suppose now that the Clifford bundle S is graded. Then:*

a) if the dimension n is even,

$$\operatorname{ind} D^+ = \frac{1}{(4\pi)^{n/2}} \int_M \operatorname{str}_x \Theta_{n/2}(x,x)dx;$$

b) if n is odd,

$$\operatorname{ind} D^+ = 0.$$

Proof. By McKean-Singer formula, $\operatorname{ind} D^+ = \operatorname{str} e^{-tD^2}$. From Proposition 0.16 we now get an asymptotic expansion

$$\operatorname{ind} D^+ = \operatorname{str} e^{-tD^2} \sim \frac{1}{(4\pi t)^{-n/2}} \sum_{k\geq 0} t^k \int_M \operatorname{str}_x \Theta_k(x,x)dx.$$

Since the function on the left is constant in t, the coefficients in the asymptotic expansion must be 0 except for the case $k = n/2$, which can only arise when n is even. □

0.2.3 The index formula for Dirac operators

From now on, we will concentrate on the nontrivial case where the dimension n is even. M^n is then a closed Riemannian manifold of even dimension, equipped with a graded Clifford bundle S with associated Dirac operator D. We will moreover require that M is *orientable*.

If we rescale and contract the coefficients of the asymptotic expansion by setting

$$\alpha_k(x) := \frac{1}{(4\pi t)^{n/2}} \operatorname{str}_x \Theta_{k+n/2}(x,x),$$

we can rewrite the previous results as

$$\operatorname{str}_x \mathcal{E}(x,x) \sim \sum_{k\geq -n/2} \alpha_k(x) t^k \tag{0.19}$$

and

$$\operatorname{ind} D^+ = \int_M \alpha_0(x)\operatorname{vol}(x),$$

where vol is the volume form on M. It is possible to characterise the differential form $\alpha_0 \operatorname{vol}$ in a geometrically significant way. This study concludes the proof of the Atiyah-Singer index theorem, but we will not explore it in any detail. For the definitions involved in the next proposition and for its proof, we address the reader to [21].

Theorem 0.20. (Local index theorem) *Let M^n be a closed Riemannian manifold of even dimension n. Suppose that M is equipped with a graded Clifford bundle S, with associated Dirac operator D. Then, the constant term $\alpha_0 \operatorname{vol}$ in the asymptotic expansion of the heat kernel $\operatorname{str}_x \mathcal{E}(t; x, x) \operatorname{vol}(x)$ as $t \to 0$ is the n-form part of*

$$\widehat{A}(M) \operatorname{ch}_s(S/\Phi),$$

where $\widehat{A}(M)$ is the $\widehat{A}$-genus in the Pontryagin classes of the Levi-Civita connection, and $\operatorname{ch}_s(S/\Phi)$ is the relative Chern character *of the Clifford bundle S.*

Theorem 0.21. (Atiyah-Singer index theorem) *Under the same hypotheses, the index of the positive part of the Dirac operator is given by*

$$\operatorname{ind} D^+ = \int_M \widehat{A}(M) \operatorname{ch}_s(S/\Phi).$$

0.2.4 Application: the Hirzebruch signature theorem

The Atiyah-Singer index theorem applies in the case of the signature operator to give an easy proof of classical Hirzebruch's signature theorem.

Let M^{4k} a closed oriented Riemannian manifold of dimension $4k$, and let $D = d + \delta$ the signature operator on M. Thanks to Proposition 0.13, the signature of M equals the index of D^+. Using the index formula of Theorem 0.21, we can hence write

$$\operatorname{sign} M = \int_M \widehat{A}(M) \operatorname{ch}_s(\Lambda^*(T^*M)/\Phi).$$

It is not hard to show that the integrand can be rewritten in the following manner. For the defnitions and proof, we refer once again to [21].

Proposition 0.22. *Let M an oriented Riemannian manifold (not necessarily closed). Then,*

$$\widehat{A}(M) \operatorname{ch}_s(\Lambda^*(T^*M)/\Phi) = L(M),$$

where $L(M)$ is the Hirzebruch L-polynomial in the Pontryagin classes of the Levi-Civita connection of M.

As a consequence, we get

Theorem 0.23. (Hirzebruch signature theorem*) Let M^{4k} a closed oriented Riemannian manifold of dimension $4k$. Then, the signature of M is given by*

$$\operatorname{sign} M = \int_M L(M) = \langle L(M), [M] \rangle.$$

0.2.5 Application: spin manifolds and positive scalar curvature

If M is an even dimensional spin manifold, with Clifford bundle $S = \$(M)$, then $ch_s(S/\Phi) = 1$ and hence the index formula becomes

$$\operatorname{ind} D\!\!\!/^+ = \int_M \hat{A}(M) = \langle \hat{A}(M), [M] \rangle. \tag{0.20}$$

Note that the $\hat{A}$-genus $\langle \hat{A}(M), [M] \rangle$ can only be nonzero when the dimension of M is multiple of 4.

As a consequence of formula (0.20) we see that the index of $D\!\!\!/^+$ is an oriented topological invariant of M: it does not depend either on the differential structure, Riemannian metric and spin structure chosen. At the same time, (0.20) also implies that the $\hat{A}$-genus of a spin manifold is always an integer. This fact was already known before the index theorem was proven, and it is reported as an important motivation to its discovery.

Combining (0.20) with the Bochner argument applied to Weitzenböck formula (0.8), we get the following result.

Theorem 0.24. (Lichnerowicz) *Let M^{4k} be a closed spin manifold. If M admits a metric of strictly positive scalar curvature, then the $\hat{A}$-genus $\langle \hat{A}(M), [M] \rangle$ vanishes.*

Theorem 0.24 says that, on spin manifolds, the $\hat{A}$-genus is an obstruction to the existence of a metric of positive scalar curvature. It is just the first of a series of results an conjectures relating the geometry of spin manifolds with the study of metrics of positive scalar curvature. For a survey on the topic, see [25] or [23].

Chapter 1

Eta invariants and the Atiyah-Patodi-Singer index theorem

An obvious consequence of classical Hirzebruch's formula is that the signature of a manifold is multiplicative under finite coverings. The same fact is not true for manifolds with boundary, as pointed out in [3] and [4]. Hence, the signature of a manifold with boundary cannot be given by the integral of a local quantity. In effect, Atiyah, Patodi and Singer proved a generalised signature formula in which a correction term appears. A surprising fact is that this correction term is a *spectral* invariant. It is called the eta invariant and depends only on the restriction of the signature operator on the boundary.

Atiyah, Patodi and Singer proved their signature formula as a nontrivial consequence of a new index theorem for manifolds with boundary, published in the series of papers [3], [4] and [5]. In order to compare the signature of a manifold with boundary with the index of the signature operator, they had to impose on the operator nonlocal boundary condition of spectral nature. The use of these global boundary conditions is among the main novelties in the Atiyah-Patodi-Singer approach to the index theory for manifolds with boundary. The choice for global boundary conditions is due to an intrinsic problem: whenever the boundary operator is nonzero, classical boundary conditions do not allow to define a finite index for the Dirac operator. This is the case for the signature operator, whose index cannot be studied by other previously known methods.

Taken alone, the eta invariant is not a topological or differential invariant of the boundary, since it depends on the metric. In the next chapters, however, we will see how a difference of eta invariants (the ρ invariant) turns out to be a useful topological invariant.

We will cover the material of [3], where the authors prove the main theorem and its application to the signature problem were proven. Another useful reference is [7], where the Atiyah-Patodi-Singer index theorem is proved in full detail.

1.1 The index theorem for manifolds with boundary

Let $M = M^{2l}$ a compact Riemannian manifold with boundary of even dimension $2l$, and let $N = N^{2l-1}$ be its boundary. $N = \partial M$ is thus a closed manifold of odd dimension $2l - 1$. First of all, we put a restriction on the geometry of M near its boundary. We will always suppose that a neighbourhood of N is isometrically diffeomorphic to a product $N \times [0, 1]$, with which we will identify it.

Let now $S = S^+ \oplus S^-$ be a graded Clifford bundle on M, with associated Dirac operator D. We will also suppose that the Hermitian structure on S is a product in the collar $N \times [0, 1]$. Thanks to these geometric assumptions, we have a useful splitting of the Dirac operator.

Proposition 1.1. *On $N \times [0, 1]$, the positive part of the Dirac operator takes the special form*

$$D^+ = \sigma \left(\frac{\partial}{\partial u} + B \right), \tag{1.1}$$

where:

- *$\sigma : S^+ \to S^-$ is the bundle local isomorphism given by Clifford multiplication by the unit inward normal vector;*
- *$\frac{\partial}{\partial u} : C^\infty(M, S^+) \to C^\infty(M, S^+)$ is the covariant derivative along the unit inward normal vector;*
- *$B : C^\infty(M, S^+) \to C^\infty(M, S^+)$ is tangential and does not depend on the normal coordinate, i.e. if $s_1, s_2 \in C^\infty(M, S^+)$ are such that $s_1(\cdot, u_1) = s_2(\cdot, u_2)$ for some $u_1, u_2 \in [0, 1]$, then $(Bs_1)(\cdot, u_1) = (Bs_2)(\cdot, u_2)$.*

Proof. Choose a local orthonormal basis of vector fields $e_1, ..., e_{2l}$, with e_{2l} pointing towards the normal direction and hence $e_1, ...e_{2l-1} \in C^\infty(M, TN)$. By definition of the Dirac operator, we locally have

$$D^+ f = \sum_{i=1}^{2l} c(e_i) \nabla_i f, \tag{1.2}$$

where c is the Clifford action and ∇_i stands for ∇_{e_i}. If we define B to be the composition $B := \sigma^{-1} D - \frac{\partial}{\partial u}$, we thus have

$$Bs = -c(e_{2l}) \sum_{i=1}^{2l} c(e_i) \nabla_i f - \nabla_{2l} f = -c(e_{2l}) \sum_{i=1}^{2l-1} c(e_i) \nabla_i f, \tag{1.3}$$

which clearly is tangential and independent on $u \in [0, 1]$. □

In fact we will view B as an operator

$$B : C^\infty(N, S^+|_N) \to C^\infty(N, S^+|_N)$$

on N. It is the Dirac operator associated to the Clifford bundle structure on $S^+|_N$ given by $\tilde{c}(v)s := -c(e_{2l})c(v)s$. In particular, it is elliptic and essentially self-adjoint. By Theorem 0.6, we know that B admits a discrete spectrum of real eigenvalues λ, with finite-dimensional eigenspaces H_λ made of smooth sections such that $L^2(N, S^+|_N) = \bigoplus_\lambda H_\lambda$. We define

$$P : C^\infty(N, S^+|_N) \to C^\infty(N, S^+|_N) \tag{1.4}$$

to be the restriction of the orthogonal projection onto $\bigoplus_{\lambda \geq 0} H_\lambda$. In order to study the index of D^+, we impose the global boundary conditions

$$P(f|_N) = 0. \tag{1.5}$$

If we define $C^\infty(M, S^+; P)$ to be the subspace of sections f satisfying the boundary conditions, this means that we are seeing D^+ as an operator

$$D^+ : C^\infty(M, S^+; P) \to C^\infty(M, S^-) \tag{1.6}$$

We now define η to be the function of one complex variable

$$\eta(s) = \sum_{\lambda \neq 0} \operatorname{sgn} \lambda \, |\lambda|^{-s}. \tag{1.7}$$

η is a well defined analytical function for $\operatorname{Re} s \gg 0$. Its value at $s = 0$, if it exists, represents heuristically a difference between the number of positive eigenvalues and the number of negative ones (that is, the *spectral asymmetry* of B). We will show that η admits a meromorphic continuation to the whole complex plane, with finite value at $s = 0$.

The main result of this section is to show that the operator (1.6) has a well defined index, which is computed as

$$\operatorname{ind} D^+ = \int_M AS + \frac{h(B) + \eta(B)}{2},$$

where $h(B)$ denotes the dimension of the null-space of B and $\eta(B) := \eta(0)$. This formula will be established by heat kernel methods. Namely, we will construct an approximate heat kernel on M by glueing together the heat kernel on the cylinder $C = N \times \mathbb{R}_+$ (with boundary conditions (1.5)) with the heat kernel on the compact double $\widehat{M}$ of M.

1.1.1 Computations on the cylinder

Consider the half cylinder

$$C = N \times \mathbb{R}_+.$$

On C it is defined the pullback bundle

$$E := p^* S^+|_N,$$

p being the product projection on N. There is an isomorphism

$$L^2(C,E) \cong L^2(\mathbb{R}_+) \otimes L^2(N, S^+|_N) = \bigoplus_\lambda L^2(\mathbb{R}_+) \otimes H_\lambda, \tag{1.8}$$

where $H_\lambda \subseteq L^2(N, S^+|_N)$ is, as above, the λ-eigenspace of B. It is useful for the following to define

$$L_0 := L^2(\mathbb{R}_+) \otimes H_0$$

and

$$L_{\neq 0} := \bigoplus_{\lambda \neq 0} L^2(\mathbb{R}_+) \otimes H_\lambda,$$

so that

$$L^2(C,E) = L_0 \oplus L_{\neq 0}. \tag{1.9}$$

Let us fix a on orthonormal base $\{\varphi_\lambda\}$ of $L^2(N, S^+|_N)$, made of eigensections $\varphi_\lambda \in H_\lambda$ (this is a slight abuse of notation: of course for each λ there will be a number of φ_λ's equal to the multiplicity of λ). If $f \in L^2(C,E)$, (1.8) says that we can write

$$f(x,u) = \sum_\lambda f_\lambda(u)\varphi_\lambda(x), \tag{1.10}$$

with $f_\lambda \in L^2(\mathbb{R}_+)$. We have then

$$\|f\|^2_{L^2} = \sum_\lambda \|f_\lambda\|^2_{L^2}.$$

Consider the *normalised Dirac operators* (acting on $C^\infty(C,E)$)

$$A := \frac{\partial}{\partial u} + B$$

Let P be as in (1.4). We impose on A the boundary condition $P(f|_N) = 0$. Let

$$C^\infty(C,E;P) := \{f \in C^\infty(C,E) \mid P(f|_N) = 0\}.$$

the space of smooth sections satisfying these boundary conditions. A similar notation will be adopted for compactly supported sections $C^\infty_c(C,E;P) \subseteq C^\infty_c(C,E)$.

On the noncompact manifold C, we define Sobolev spaces $W^r(C,E)$ as the closure of $C^\infty_c(C,E)$ in the norm

$$\|f\|^2_{W^r} = \|f\|_{L^2} + \left\|\frac{\partial^r f}{\partial u^r}\right\|^2_{L^2} + \|Bf\|_{L^2} + \cdots \|B^r f\|_{L^2}.$$

Basic results as the Sobolev embedding theorem keeps holding in this situation. Thanks to the following general result of analysis on manifolds with boundary (which is proved for example in [7]), it also makes sense to speak of the space $W^r(C,E;P)$ for $r \geq 1$.

Proposition 1.2. (Trace theorem) *The restriction to the boundary*

$$\begin{aligned} C_c^\infty(C,E) &\to C^\infty(N, S^+|_N) \\ f &\mapsto f|_N \end{aligned}$$

extends to a bounded operator

$$W^1(C,E) \to L^2(N, S^+|_N).$$

Alongside A, we will study the operator

$$A^* := -\frac{\partial}{\partial u} + B.$$

A^* is a formal adjoint of A: that is, we have $(Af,g)_{L^2} = (f, A^*g)_{L^2}$ for each f,g with compact support contained in the interior of the cylinder $C = N \times \mathbb{R}_+$. We will impose on A^* the *adjoint boundary condition* $(1-P)(f|_N) = 0$. As for P, we define the spaces of functions $C^\infty(C,E;1-P)$, $C_c^\infty(C,E;1-P)$ and $W^r(C,E;1-P)$ $(r \geq 1)$. It is easy to verify, by integration by parts, that we have now

$$\begin{gathered} (Af,g)_{L^2} = (f, A^*g)_{L^2} \\ \forall\, f \in C_c^\infty(C,E;P),\ g \in C_c^\infty(C,E;1-P). \end{gathered} \tag{1.11}$$

This is stronger than formal self-adjointeness, since the functions are here allowed to be nonzero on the boundary. One of the goals of this subsection is to prove that the closures of these operators are actually mutually adjoint in the stronger sense of unbounded operator theory.

As a first step, we shall now construct a fundamental solution for the equation $Df = g$, with $g \in C_c^\infty(C,E)$.

Proposition 1.3. *There exists a linear operator*

$$Q : C_c^\infty(C,E) \to C^\infty(C,E;P)$$

such that:

(i) $AQg = g$ *for all* $g \in C_c^\infty(C,E)$*;*

(ii) $QAf = f$ *for all* $f \in C_c^\infty(C,E;P)$*.*

(iii) The restriction of Q *to* $C_c^\infty(C,E) \cap L_{\neq 0}$ *extends to a bounded linear operator*

$$Q : W^r(C,E) \cap L_{\neq 0} \to W^{r+1}(C,E;P) \cap L_{\neq 0}$$

for each r*.*

Proof. For each $g \in C_c^\infty(C,E)$, we want to define $Qg \in C^\infty(C,E;P)$ such that (i) and (ii) are verified. As in (1.10), we can write

$$g(x,u) = \sum_\lambda g_\lambda(u)\varphi_\lambda(x).$$

Suppose now that Qg is defined. Even if it is not square integrable, we can still write it as

$$(Qg)(x,u) = \sum_\lambda h_\lambda(u)\varphi_\lambda(x)$$

with h_λ's smooth functions on $\mathbb{R}_+$. Suppose that (i) is verified. We then get, for each λm

$$\left(\frac{d}{du} + \lambda\right) h_\lambda = g_\lambda \tag{1.12}$$

for each λ. This is a family of ordinary differential equations, with solutions

$$h_\lambda(u) = k_\lambda e^{-\lambda u} + \int_0^u e^{\lambda(v-u)} g_\lambda(v)dv. \tag{1.13}$$

In order that Qg satisfies the boundary condition and that (ii) holds as well, we need to choose

$$\begin{cases} k_\lambda = 0, & \text{for } \lambda \geq 0 \\ k_\lambda = -\displaystyle\int_0^\infty e^{\lambda v} g_\lambda(v)dv, & \text{for } \lambda < 0, \end{cases} \tag{1.14}$$

so that

$$h_\lambda(u) = \begin{cases} \displaystyle\int_0^u e^{\lambda(v-u)} g_\lambda(v)dv, & \text{for } \lambda \geq 0 \\ \displaystyle\int_u^\infty e^{\lambda(v-u)} g_\lambda(v)dv, & \text{for } \lambda < 0. \end{cases} \tag{1.15}$$

The operator Q is hence necessarily given by $Qg := \sum_\lambda h_\lambda \varphi_\lambda$, with h_λ given by (1.15). We need to prove that Qg is a well defined smooth section. This will follow from point (iii). Indeed, (iii) implies that $Qg \in W^s(C,E;P)$ for each $s \in \mathbb{N}$, whenever $g \in C_c^\infty(C,E) \cap L_{\neq 0}$. By Sobolev embedding theorem, then, $Qg \in C^\infty(C,E;P)$. The assertion easily follows also for any $g \in C_c^\infty(C,E)$.

We prove hence point (iii), by estimating the Sobolev norms of Qg. Equation (1.13) is easily rewritten in terms of the Fourier-Laplace transform $\widetilde{f}(\xi) = \int_0^\infty e^{-i\xi t} f(u)$ as

$$\widetilde{h}_\lambda(\xi) = \frac{\widetilde{g}_\lambda(\xi) + k_\lambda}{i\xi + \lambda}.$$

We will denote by $\|\cdot\|$ the norms in $L^2(\mathbb{R})$ and $L^2(\mathbb{R}_+)$. For $\lambda \geq 0$, we have $k_\lambda = 0$ and hence

$$\lambda \|\widetilde{h}_\lambda\| \leq \|\widetilde{g}_\lambda\| \quad (\text{for } \lambda \geq 0). \tag{1.16}$$

For $\lambda < 0$, we use the elementary inequality $|a+b|^2 \leq 2(|a|^2 + |b|^2)$ to get

$$\|\widetilde{h}_\lambda\|^2 \leq 2\left(\frac{\|\widetilde{g}_\lambda\|^2}{\lambda^2} + |k_\lambda|^2 \int_{\mathbb{R}} \frac{d\xi}{\xi^2 + \lambda^2}\right) = 2\left(\frac{\|\widetilde{g}_\lambda\|^2}{\lambda^2} + \frac{|k_\lambda|^2 \pi}{|\lambda|}\right). \tag{1.17}$$

In this case k_λ is given by $-\int_0^\infty e^{\lambda v} g_\lambda(v)dv$ and hence

$$|k_\lambda|^2 \leq \left(\int_0^\infty |e^{\lambda v} g_\lambda(v)|dv\right)^2 \leq \|e^{\lambda v}\|^2 \|g_\lambda\|^2 = \frac{1}{2|\lambda|}\frac{\|\widetilde{g}_\lambda\|^2}{2\pi}$$

(in the last equality, we have explicitly computed the first term and applied Parseval's formula to the second). Putting this into (1.17) and taking the square root, we get

$$|\lambda|\|\widetilde{h}_\lambda\| \le 2\|\widetilde{g}_\lambda\| \quad (\text{for } \lambda < 0). \tag{1.18}$$

Inequalities (1.16) and (1.18) are also valid on h_λ and g_λ, thanks to Parseval's formula. Putting them together, we obtain

$$|\lambda|\|h_\lambda\| \le 2\|g_\lambda\| \quad \text{for any } \lambda. \tag{1.19}$$

This only gives continuity of $Q : L_{\neq 0} \to L^2(C, E; P)$. However, from (1.13) and (1.19) we also get

$$\left\| \frac{dh_\lambda}{du} \right\| \le \|g_\lambda\| \quad \text{for any } \lambda.$$

The W^1-norm of f is

$$\|f\|^2_{W^1} = \|f\|^2_{L^2} + \left\| \frac{\partial f}{\partial u} \right\|^2_{L^2} + \|Bf\|^2_{L^2} = \sum_\lambda \left((1+\lambda^2)\|f^2_\lambda\| + \left\| \frac{df_\lambda}{du} \right\|^2 \right).$$

As a consequence, we have continuity $Q : L_{\neq 0} \to W^1(C, E)$. In effect, suppose that $g_0 = 0$. Then

$$\|Qg\|^2_{W^1} = \sum_{\lambda \neq 0} \left((1+\lambda^2)\|h_\lambda\|^2 + \left\| \frac{dh_\lambda}{du} \right\|^2 \right) \le \sum_{\lambda \neq 0} (\frac{2}{\lambda^2} + 4 + 9)\|g_\lambda\|^2 \le C\|g\|^2_{L^2}.$$

Moreover, Qg satisfies the boundary condition $P((Qg)|_N) = 0$, thanks to Proposition 1.2. Hence, $Qg \in W^1(C, E; P) \cap L_{\neq 0}$. This establishes (iii) for $r = 0$. Differentiating r times the relation (1.12) with respect to u, we get continuity of $Q : W^r(C, E) \cap L_{\neq 0} \to W^{r+1}(C, E; P)$ in a similar way. □

Remark 1.4. *In a completely anologous manner, we can construct a fundamental solution Q^* for the operator A^* (with boundary condition $1 - P$), sharing all the properties proved for Q in Proposition 1.3.*

We now regard A and A^* as unbounded operators on $L^2(C, E)$, with domains $C^\infty_c(C, E; P)$ and $C^\infty_c(C, E; 1 - P)$ respectively. We call $\mathcal{A}$ and $\mathcal{A}^*$ the Hilbert closures of A and A^* respectively. Recalling the decomposition $L^2(C, E) = L_0 \oplus L_{\neq 0}$ of (1.9), we see that the domains of A and A split accordingly. Moreover these operators preserve the decomposition, sending each summand into itself. This means that the operators A and A^* split. By definition of closure,the operators $\mathcal{A}$ and $\mathcal{A}^*$ split as well. The next results are crucial for the following.

Lemma 1.5. *The operator Q of Proposition 1.3 is a bounded inverse of $\mathcal{A}|_{L_{\neq 0}}$. Similarly, the operator Q^* of the Remark 1.4 is a bounded inverse of $\mathcal{A}^*|_{L_{\neq 0}}$.*

Proof. We will prove that $Q\mathcal{A}f = f$ for any $f \in \mathrm{Dom}(\mathcal{A}) \cap L_{\neq 0}$, and $\mathcal{A}Qg = g$ for any $g \in L_{\neq 0}$. The proof is analogous for $\mathcal{A}^*$ and Q^*.

Let $f \in \mathrm{Dom}(\mathcal{A}) \cap L_{\neq 0}$. Since $\mathcal{A}$ is the closure of A, by definition there is a sequence $\{f_n\} \subseteq C_c^\infty(C, E; P) \cap L_{\neq 0}$ such that $f_n \to f$ and $Af_n \to \mathcal{A}f$ in L^2. By the Proposition, we have $f_n = QAf_n$ and by continuity of Q this must converge to $Q\mathcal{A}f$. By uniqueness of the limit, then, $Q\mathcal{A}f = f$.

Let now $g \in L_{\neq 0}$. By density, we can find a sequence $\{g_n\} \subseteq C_c^\infty(C, E) \cap L_{\neq 0}$ such that $g_n \to g$ in L^2. Then $\{Qg_n\} \subseteq C_c^\infty(C, E; P)$, so that $AQg_n = g_n$. By continuity we have $Qg_n \to Qg$, and moreover $AQg_n = g_n \to g$. Hence, by definition of the closure, $Qg \in \mathrm{Dom}(\mathcal{A})$ and $\mathcal{A}Qg = g$. □

Proposition 1.6. *$\mathcal{A}$ and $\mathcal{A}^*$ are mutually adjoint.*

Proof. Since this operators split on $L^2(C, E) = L_0 \oplus L_{\neq o}$, we can check adjointness separately on the two summands. On L_0 we have $\mathcal{A} = \partial/\partial t$ and $\mathcal{A}^* = -\partial/\partial t$, so adjointness is clear. On $L_{\neq 0}$, Lemma 1.5 gives us bounded inverses Q, Q^* of $\mathcal{A}$, $\mathcal{A}^*$. Let us show that these operators are mutually adjoint. For any $f, g \in C_c^\infty(C, E)$, recalling property (1.11) we have

$$(Qf, g)_{L^2} = (Qf, A^*Q^*g)_{L^2} \overset{(1.11)}{=} (AQf, Q^*g)_{L^2} = (f, Q^*g)_{L^2}.$$

By continuity, this must be true in the whole space of L^2 sections. The thesis is now a consequence of the general (and elementary) result in unbounded operator theory that adjoints commute with inverses. □

Corollary 1.7. $\mathrm{Dom}(\mathcal{A}) = W^1(C, E; P)$*,* $\mathrm{Dom}(\mathcal{A}^*) = W^1(C, E; 1 - P)$*.*

Proof. The inclusion $\mathrm{Dom}(\mathcal{A}) \subseteq W^1(C, E; P)$ follows by definition of closure, since convergence in L^2 of $\{f_n\}$ and $\{Af_n\}$ (for $\{f_n\} \subseteq C_c^\infty(C, E; P)$) is equivalent to convergence in W^1. Similarly, we have $\mathrm{Dom}(\mathcal{A}^*) \subseteq W^1(C, E; 1 - P)$.

Let now $f \in W^1(C, E, ; P)$. We want to show that $f \in \mathrm{Dom}(\mathcal{A})$. Thanks to Proposition 1.6, $\mathcal{A}$ is the adjoint of $\mathcal{A}^*$. In particular

$$\mathrm{Dom}(\mathcal{A}) = \{h \in L^2(C, E) \,|\, \forall\, g \in \mathrm{Dom}(\mathcal{A}^*)\, \exists\, u \text{ s.t. } (\mathcal{A}^*g, h)_{L^2} = (g, u)_{L^2}\}.$$

Let then $g \in \mathrm{Dom}(\mathcal{A}^*) \subseteq W^1(C, E; 1-P)$. f and g are approximated in W^1 by sequences $\{f_n\} \subseteq C_c^\infty(C, E; P)$ and $\{g_n\} \subseteq C_c^\infty(C, E; 1 - P)$ respectively. Let u the limit in L^2 of $\{Af_n\}$. Recalling the property (1.11), we have

$$(\mathcal{A}^*g, f)_{L^2} = \lim_n\, (A^*g_n, f_n)_{L^2} \overset{(1.11)}{=} \lim_n\, (g_n, Af_n)_{L^2} = (g, u)_{L^2}.$$

Hence, $f \in \mathrm{Dom}(\mathcal{A})$ and consequently $W^1(C, E; P) \subseteq \mathrm{Dom}(\mathcal{A})$. Similarly, $W^1(C, E; 1 - P) \subseteq \mathrm{Dom}(\mathcal{A}^*)$ □

Corollary 1.8. *The operators $\mathcal{A}^*\mathcal{A}$ and $\mathcal{A}\mathcal{A}^*$ are self-adjoint. They both act as $-\partial/\partial u + B^2$ and their domains are given by:*

$$\mathrm{Dom}(\mathcal{A}^*\mathcal{A}) = \{f \in W^2(C, E) \,|\, P(f|_N) = 0,\ (1 - P)((\mathcal{A}f)|_N) = 0\};$$

$$\mathrm{Dom}(\mathcal{A}\mathcal{A}^*) = \{f \in W^2(C, E) \,|\, (1 - P)(f|_N) = 0,\ P((\mathcal{A}^*f)|_N) = 0\}.$$

Proof. It is a general fact in unbounded operator theory that the product of two mutually adjoint operators is densely defined and self-adjoint (see [10]). The expressions for the domains of $\mathcal{A}^*\mathcal{A}$ and $\mathcal{A}\mathcal{A}^*$ easily follow from Corollary 1.7. □

Since $\mathcal{A}^*\mathcal{A}$ and $\mathcal{A}\mathcal{A}^*$ are positive self-adjoint operators, by the functional calculus we can define, for any "time" $t \geq 0$, the *heat operators* $e^{-t\mathcal{A}^*\mathcal{A}}$ and $e^{-t\mathcal{A}\mathcal{A}^*}$. These are bounded operators on $L^2(C, E)$ and as in the compact case they are fundamental solutions to the *heat equations* on the cylinder

$$\left(\frac{\partial}{\partial t} + \mathcal{A}^*\mathcal{A}\right) f = 0, \tag{1.20}$$

$$\left(\frac{\partial}{\partial t} + \mathcal{A}\mathcal{A}^*\right) f = 0. \tag{1.21}$$

Let $\mathcal{E}_c$ and $\mathcal{E}_{c*}$ the Schwartz kernels associated to $e^{-t\mathcal{A}^*\mathcal{A}}$ and $e^{-t\mathcal{A}\mathcal{A}^*}$ respectively, called the *cylindrical heat kernels.* These are time-dependent sections of the bundle $E \boxtimes E^*$ over $C \times C$. Hence, they are function of $\mathbb{R}_+ \times C \times C = \mathbb{R}_+ \times (N \times \mathbb{R}_+) \times (N \times \mathbb{R}_+)$. As we did in the compact case, we want to study the asymptotic behaviour of the heat kernels for $t \to 0$. First of all, we apply separation of variables again to give a diagonalised expression for $\mathcal{E}_c$ and $\mathcal{E}_{c*}$.

Lemma 1.9.

$$\mathcal{E}_c(t; x, u, y, v) = \sum_\lambda e_\lambda(t; u, v)\varphi_\lambda(x) \otimes \varphi_\lambda(y)^*. \tag{1.22}$$

with

$$e_\lambda(t; u, v) = \begin{cases} \dfrac{e^{-\lambda^2 t}}{\sqrt{4\pi t}} \left(e^{-\frac{(u-v)^2}{4t}} - e^{-\frac{(u+v)^2}{4t}}\right) & \text{if } \lambda \geq 0 \\ \dfrac{e^{-\lambda^2 t}}{\sqrt{4\pi t}} \left(e^{-\frac{(u-v)^2}{4t}} + e^{-\frac{(u+v)^2}{4t}}\right) + \lambda e^{-\lambda(u+v)} \operatorname{erfc}\left(\dfrac{u+v}{2\sqrt{t}} - \lambda\sqrt{t}\right) & \text{if } \lambda < 0, \end{cases}$$

where erfc *is the complementary error function defined by*

$$\operatorname{erfc}(s) = \frac{2}{\sqrt{\pi}} \int_s^\infty e^{-\xi^2} d\xi.$$

Similarly,

$$\mathcal{E}_{c*}(t; x, u, y, v) = \sum_\lambda e_{*\lambda}(t; u, v)\varphi_\lambda(x) \otimes \varphi_\lambda(y)^*. \tag{1.23}$$

with

$$e_{*\lambda}(t; u, v) = \begin{cases} \dfrac{e^{-\lambda^2 t}}{\sqrt{4\pi t}} \left(e^{-\frac{(u-v)^2}{4t}} + e^{-\frac{(u+v)^2}{4t}}\right) - \lambda e^{\lambda(u+v)} \operatorname{erfc}\left(\dfrac{u+v}{2\sqrt{t}} + \lambda\sqrt{t}\right) & \text{if } \lambda \geq 0 \\ \dfrac{e^{-\lambda^2 t}}{\sqrt{4\pi t}} \left(e^{-\frac{(u-v)^2}{4t}} - e^{-\frac{(u+v)^2}{4t}}\right) & \text{if } \lambda < 0. \end{cases}$$

In particular, the cylindrical heat kernels are smooth.

Proof. Suppose $f(t;x,u) = \sum_\lambda f_\lambda(t;u)\varphi_\lambda(x)$. Equation (1.20) is then rewritten as the family of equations

$$\left(\frac{\partial}{\partial t} - \frac{\partial^2}{\partial u^2} + \lambda^2\right) f_\lambda = 0,$$

with boundary conditions

$$\begin{cases} f_\lambda(t;0) = 0, & \text{if } \lambda \geq 0 \\ \left(\dfrac{\partial f_\lambda}{\partial u} + \lambda f_\lambda\right)(t;0) = 0 & \text{if } \lambda < 0, \end{cases}$$

given by the domain of $\mathcal{A}^*\mathcal{A} = -\frac{\partial^2}{\partial u^2} + \lambda^2$. A fundamental solution for this problem can be seen to have kernel equal to e_λ as given in the statement of this lemma. Multiplying by $\varphi_\lambda \otimes \varphi_\lambda^*$ and summing over the spectrum of B we must clearly obtain the kernel of $e^{-t\mathcal{A}^*\mathcal{A}}$, which is a fundamental solution to (1.20). □

We will now use these formulas to show that the value of the heat kernels outside the diagonal decays exponentially as $t \to 0$.

Proposition 1.10. *There exist constants $K, K_* > 0$ such that, if t is close enough to* 0*,*

$$|\mathcal{E}_c(t;x,u,y,v)| \leq Kt^{-\frac{n}{2}} e^{-\frac{(u-v)^2}{4t}} \tag{1.24}$$

and

$$|\mathcal{E}_{c*}(t;x,u,y,v)| \leq K_* t^{-\frac{n}{2}} e^{-\frac{(u-v)^2}{4t}}. \tag{1.25}$$

Proof. We will show (1.24), (1.25) being similar. Let e_λ as in Lemma 1.9. Computations show that, for any λ,

$$|e_\lambda(t;u,v)| \leq \frac{3}{\sqrt{\pi t}} e^{-\frac{\lambda^2 t}{2}} e^{-\frac{(u-v)^2}{4t}}$$

(this is clear for $\lambda \geq 0$; for $\lambda < 0$ we use the inequalities $\operatorname{erfc}(s) \leq e^{-s^2}$ and $|\lambda| \leq \frac{e^{\lambda^2 t}}{\sqrt{t}}$). Thanks to formula (1.22) we get thus

$$\begin{aligned} |\mathcal{E}_{c*}(t;x,u,y,v)| &\leq \frac{3}{\sqrt{\pi t}} \left(\sum_\lambda e^{-\frac{\lambda^2 t}{2}} |\varphi_\lambda(x)||\varphi_\lambda(y)|\right) e^{-\frac{(u-v)^2}{4t}} \leq \\ &\leq \frac{3}{2\sqrt{\pi t}} \left(\sum_\lambda e^{-\frac{\lambda^2 t}{2}} |\varphi_\lambda(x)|^2 + \sum_\lambda e^{-\frac{\lambda^2 t}{2}} |\varphi_\lambda(y)|^2\right) e^{-\frac{(u-v)^2}{4t}}. \end{aligned}$$

But $\sum_\lambda e^{-\frac{\lambda^2 t}{2}} |\varphi_\lambda(x)|^2$ is the fibre trace of the heat kernel on the diagonal for the Dirac operator $B/\sqrt{2}$, which thanks to Proposition 0.18 is bounded by some $K' t^{-\frac{2l-1}{2}}$ as $t \to 0$. As a consequence, we get the desired inequality. □

We now want to study the contribution from the diagonal of the heat kernels. It turns out that much of the information we will need in the following is contained in the symmetric integral

$$\mathcal{K}(t) := \int_0^\infty \int_N \operatorname{tr}_{(y,v)}(\mathcal{E}_c(t;y,v,y,v) - \mathcal{E}_{c*}(t;y,v,y,v))dydv.$$

Proposition 1.11. *(i) With the convention of* $\operatorname{sgn}(0) = 1$*, we have*

$$\mathcal{K}(t) = -\frac{1}{2}\sum_\lambda \operatorname{sgn}\lambda \operatorname{erfc}(|\lambda|\sqrt{t}). \tag{1.26}$$

(ii) For $\operatorname{Re} s > n/2$*, where* $n := 2l-1$*, we have*

$$\int_0^\infty \left(\mathcal{K}(t) + \frac{h(B)}{2}\right) t^{s-1}dt = -\frac{\Gamma\left(s+\frac{1}{2}\right)}{2s\sqrt{\pi}}\eta(2s), \tag{1.27}$$

where

- $h(B) := \dim\ker B$*;*
- Γ *denotes the Euler gamma function defined by*

$$\Gamma(r) = \int_0^\infty \xi^{r-1}e^{-\xi}d\xi;$$

- $\eta(s) = \sum_{\lambda\neq 0}\operatorname{sgn}\lambda\,|\lambda|^{-s}$ *is the eta function defined in* (1.7).

Proof. (i) We apply the formulas of Lemma 1.9. Thanks to (1.22) and (1.23), from the definition of $\mathcal{K}$ we obtain

$$\begin{aligned}\mathcal{K}(t) &= \sum_\lambda \int_0^\infty \int_N (e_\lambda(t;v,v) - e_{\lambda*}(t;v,v))\varphi_\lambda(y)\otimes\varphi_\lambda(y)^*dydv = \\ &= \sum_\lambda \int_0^\infty (e_\lambda(t;v,v) - e_{\lambda*}(t;v,v))dv \int_N |\varphi_\lambda(y)|^2dy = \\ &= \sum_\lambda \int_0^\infty (e_\lambda(t;v,v) - e_{\lambda*}(t;v,v))dv.\end{aligned}$$

The explicit formulas for e_λ and $e_{\lambda*}$ give then

$$\begin{aligned}\mathcal{K}(t) &= \sum_\lambda \operatorname{sign}\lambda \int_0^\infty \left(-\frac{e^{-\lambda^2t}e^{-\frac{v^2}{t}}}{\sqrt{\pi t}} + |\lambda|e^{2|\lambda|v}\operatorname{erfc}\left(\frac{v}{\sqrt{t}} + |\lambda|\sqrt{t}\right)\right) = \\ &= \sum_\lambda \operatorname{sign}\lambda \int_0^\infty \frac{\partial}{\partial v}\left(\frac{1}{2}e^{2|\lambda|v}\operatorname{erfc}\left(\frac{v}{\sqrt{t}} + |\lambda|\sqrt{t}\right)\right) = \\ &= -\frac{1}{2}\sum_\lambda \operatorname{sgn}\lambda\operatorname{erfc}(|\lambda|\sqrt{t}).\end{aligned}$$

(ii) We can rewrite (1.26) as

$$\mathcal{K}(t) = -\frac{1}{2}h(B) - \sum_{\lambda\neq 0} \operatorname{sgn}\lambda \operatorname{erfc}(|\lambda|\sqrt{t}),$$

from which it is apparent that $\mathcal{K}(t) \to -\frac{1}{2}h(B)$ as $t \to \infty$. Moreover, since its derivative is

$$\mathcal{K}'(t) = \frac{1}{2\sqrt{\pi t}} \sum_{\lambda} \lambda e^{-\lambda^2 t}, \tag{1.28}$$

we see that this convergence is exponentially fast. This resolves the problem of convergence near $+\infty$ of the integral in (1.27). From (1.26), since $\operatorname{erfc}(r) \leq \frac{2}{\sqrt{\pi}} e^{-r^2}$, we also see that

$$|\mathcal{K}(t)| \leq \frac{1}{\sqrt{\pi}} \sum_{\lambda} e^{-\lambda^2 t}.$$

But $\sum_{\lambda} e^{-\lambda^2 t}$ is the trace of the heat operator e^{-tB^2} on the compact manifold N, hence by Proposition 0.18 it is bounded by some $Kt^{-n/2}$ as $t \to 0$. As a consequence, for $\operatorname{Re} s > n/2$, the integral in (1.27) is convergent. Integrating by parts $(\mathcal{K}(t) + h(B)/2)t^{s-1}$, we obtain (1.27). □

Corollary 1.12. *If $\mathcal{K}$ has an asymptotic expansion as $t \to 0$ given by*

$$\mathcal{K}(t) \sim \sum_{k \geq -n} a_k t^k,$$

then, for each $N \geq -n$, we can write

$$\eta(s) = -\frac{s\sqrt{\pi}}{\Gamma\left(\frac{s+1}{2}\right)} \left(\frac{h(B)}{s} + 2\sum_{k=-n}^{N} \frac{a_k}{2k+s} + \theta_N(s) \right),$$

where θ_N is a holomorphic function for $\operatorname{Re} s > -(N+1)$. This expression gives a meromorphic continuation of η to the whole complex plane. In particular, η is holomorphic near 0 with value $\eta(B) := \eta(0)$ equal to

$$\eta(B) = -(h(B) + 2a_0). \tag{1.29}$$

Proof. A rescaling of (1.27) yelds:

$$\eta(s) = -\frac{s\sqrt{\pi}}{\Gamma\left(\frac{s+1}{2}\right)} \int_0^\infty \left(\frac{h(B)}{2} + \mathcal{K}(t) \right) t^{\frac{s}{2}-1} dt.$$

Thanks to the asymptotic expansion of $\mathcal{K}$ near 0, we can write

$$\mathcal{K}(t) = \sum_{k \geq -n} a_k t^k + \mathcal{R}(t),$$

with $\mathcal{R}$ smooth on $[0,1]$. Now let $N \geq -n$. We can split the integral as:

$$\int_0^\infty \left(\frac{h(B)}{2} + \mathcal{K}(t)\right) t^{\frac{s}{2}-1} dt =$$

$$\left[\int_0^1 \left(\frac{h(B)}{2} + \sum_{k=-n}^{N} a_k t^k\right) t^{\frac{s}{2}-1} dt\right] + \left[\int_0^1 \left(\sum_{k \geq N+1} a_k t^k + \mathcal{R}(t)\right) t^{\frac{s}{2}-1} dt + \int_1^\infty\right].$$

The first summand, view as a function of s, now has a meromorphic extension to the whole complex plane equal to $\left(\frac{h(B)}{s} + 2\sum_{-n}^{N} \frac{a_k}{2k+s}\right)$, while the second is clearly holomorphic for $\operatorname{Re} s > -(N+1)$. □

We conclude the subsection with one final remark that will be useful in the following.

Remark 1.13. *If we define* $\mathcal{K}_\delta(t) := \int_0^\delta \int_N (\mathcal{E}_c(t; y, v, y, v) - \mathcal{E}_{c*}(t; y, v, y, v)) dy dv$, *integrating only between* 0 *and* δ, *its asymptotic behaviour for* $t \to 0$ *is the same of* $\mathcal{K}$. *From the computations in Proposition 1.11, we have in fact, for* $t \to 0$,

$$\begin{aligned}
|\mathcal{K}_\delta(t) - \mathcal{K}(t)| &= \left|\sum_\lambda \frac{\operatorname{sgn}\lambda}{2} e^{2|\lambda|\delta} \operatorname{erfc}(\frac{\delta}{\sqrt{t}} + |\lambda|\sqrt{t})\right| \leq \\
&\leq \frac{1}{\sqrt{\pi}} \sum_\lambda e^{2|\lambda|\delta} e^{-(\frac{\delta^2}{t} + 2|\lambda|\delta + \lambda^2 t)} = \\
&= \frac{1}{\sqrt{\pi}} e^{-\frac{\delta^2}{t}} \sum_\lambda e^{-\lambda^2 t} < K e^{-\frac{\delta^2}{t}} t^{-\frac{n}{2}}
\end{aligned}$$

and hence the difference is exponentially small.

1.1.2 Parametrix constructions

We now put ourselves back in the framework of the beginning of the section. We thus have a manifold M^{2l} with boundary N, a graded Clifford bundle S on M and, correspondently, a graded Dirac operator $D : C^\infty(M, S) \to C^\infty(M, S)$. Thanks to Proposition 1.1, on a collar $N \times [0,1] \subseteq M$ the positive part of D takes the special form

$$D^+ = \sigma\left(\frac{\partial}{\partial u} + B\right) = \sigma A,$$

where $\sigma : S^+ \to S^-$ is Clifford multiplication by the unit inward normal vector. Similarly, the negative part of D will take the form

$$D^- = \left(-\frac{\partial}{\partial u} + B\right)\sigma^{-1} = A^*\sigma^{-1}.$$

As in (1.11), the boundary conditions P and $1-P$ are such that

$$\begin{aligned} &(D^+f,g)_{L^2} = (f,D^-g)_{L^2} \\ &\forall\, f \in C_c^\infty(M,S^+;P),\ g \in C_c^\infty(M,S^-;1-P) \end{aligned} \tag{1.30}$$

One of the goals of this section is to prove that the Hilbert closures of D^+ and D^- (with boundary conditions) are mutually adjoint. Note that an analogous trace theorem to Proposition 1.2 holds in this situation, so that the spaces $W^r(M,S^+;P)$ and $W^r(M,S^-;1-P)$ can be defined.

First, we construct parametrices for D^+ and D^-. To do so, it is useful to define the *closed double* of M

$$\widehat{M} := M \cup_N M.$$

Since M is isometrically diffeomorphic to a product near N, all the structure on M is naturally extended to $\widehat{M}$. In particular, the Dirac operator D is naturally extended to a Dirac operator $\widehat{D}$ on $\widehat{M}$.

Let us now concentrate on the positive part. We will construct a parametrix for D^+ by glueing together the following two pieces:

- the parametrix Q of A of Proposition 1.3, defined on the cylinder $C = N \times [0,\infty)$;
- the parametrix $\widehat{R}^+$ of $\widehat{D}^+$ given by Proposition 0.9, defined on $\widehat{M}$.

Of course we could restrict $\widehat{R}^+$ to M: the problem with this choice is that it does not take care of the boundary condition. Hence, we will have to smooth it to 0 near the boundary, and let Q do the work on that side. On this purpose, for any $a < b \in (0,1)$ we define $\rho_{a,b}$ to be an increasing smooth function on $[0,1]$ with value 0 on $[0,a+\varepsilon]$ and 1 on $[b-\varepsilon,1]$, and then the functions:

$$\phi_1 = 1-\rho_{\frac{3}{4},1} \qquad \phi_2 = \rho_{\frac{1}{4},\frac{1}{2}} \qquad \psi_1 = 1-\rho_{\frac{1}{2},\frac{3}{4}} \qquad \psi_2 = \rho_{\frac{1}{2},\frac{3}{4}}. \tag{1.31}$$

Of course, we can extend these functions by 0, 1 to C and to $\widehat{M}$. Define now

$$R^+ := \phi_1 Q \sigma^{-1} \psi_1 + \phi_2 \widehat{R}^+ \psi_2,$$

where the ϕ_i, ψ_i are acting as multiplication operators. Thanks to the cuttings made by these functions, R^+ is a well defined operator

$$R^+ : C^\infty(M,S^-) \to C^\infty(M,S^+;P).$$

Proposition 1.14. *R^+ is a parametrix for $D^+ : C^\infty(M,S^+;P) \to C^\infty(M,S^-)$. That is, the operators D^+R^+-1 and R^+D^+-1 have smooth kernel. Moreover, R^+ can be extended to a continuous operator $R^+ : W^s(M,S^-) \to W^{s+1}(M,S^+;P)$ for any $s \in \mathbb{N}$.*

Proof. Let $\widehat{K}$ the smoothing operator on $\widehat{M}$ such that $\widehat{D}^+\widehat{R} - 1 = \widehat{K}$. Let $f \in C^\infty(M,S^-)$ and let us study $(D^+R^+-1)f$.

a) Outside the collar $N \times [0,1]$, and on $N \times [\frac{3}{4}, 1]$, we locally have

$$(D^+R^+ - 1)f = 0 + \widehat{D}^+\widehat{R}^+ f - f = \widehat{K} f = \widehat{K}\psi_2 f.$$

b) On $N \times [\frac{1}{2}, \frac{3}{4}]$, recalling that $D^+ = \sigma A$, we locally have

$$(D^+R^+ - 1)f = \sigma A Q \sigma^{-1}\psi_1 f + \widehat{D}^+\widehat{R}^+\psi_2 f - f = \psi_1 f + \psi_2 f + \widehat{K}\psi_2 f - f = \\ = \widehat{K}\psi_2 f.$$

c) On $N \times [0, \frac{1}{2}]$ we locally have

$$(D^+R^+ - 1)f = \sigma A Q \sigma^{-1} f + 0 - f = 0 = \widehat{K}\psi_2 f.$$

Globally speaking, we have then $(D^+R^+ - 1) = \widehat{K}\psi_2$, which of course has smooth kernel. Similarly, one shows that $R^+D^+ - 1$ has a smooth kernel.

Observe now that the operator Q in Proposition 1.3, when multiplied by the compactly supported function ϕ_1 on the left, becomes a continuous operator $W^s \to W^{s+1}$ (we only missed the estimate for $\lambda = 0$, that now follows from (1.15)). Since the parametrix $\widehat{R}^+$ is also continuous as an operator $W^s \to W^{s+1}$ (see Proposition 0.9), the same property for R^+ is established by its definition. □

Remark 1.15. *In a completely analogous manner, we contruct a parametrix $R^- : C^\infty(M, S^+) \to C^\infty(M, S^-; 1 - P)$ for the operator D^-, which is extended to a continuous map $R^- : W^s(M, S^+) \to W^{s+1}(M, S^-; 1 - P)$.*

Consider now their Hilbert closures $\mathcal{D}^+$ and $\mathcal{D}^-$ as unbounded operators between $L^2(M, S^+)$ and $L^2(M, S^-)$. The existence of the parametrices allow us to prove the following result.

Proposition 1.16. *The operators $\mathcal{D}^+$ and $\mathcal{D}^-$ are mutually adjoint. Their domains are given by:*

$$\mathrm{Dom}\, \mathcal{D}^+ = W^1(M, S^+; P);$$

$$\mathrm{Dom}\, \mathcal{D}^- = W^1(M, S^-; 1 - P).$$

Proof. We will prove that:

a) $\mathrm{Dom}\, \mathcal{D}^+ = W^1(M, S^+; P)$ and $\mathrm{Dom}\, \mathcal{D}^- = W^1(M, S^-; 1 - P)$;

b) $\mathrm{Dom}(\mathcal{D}^-)^* = W^1(M, S^+; P)$ and $(\mathcal{D}^-)^*$ coincides with $\mathcal{D}^+$.

a) $\mathrm{Dom}\, \mathcal{D}^+ \subseteq W^1(M, S^+; P)$ is obvious by definition of Hilbert closure and the trace theorem. Let hence $f \in W^1(M, S^+; P)$: to prove $f \in \mathrm{Dom}\, \mathcal{D}^+$ we want to approximate f in the W^1-norm by functions in $C^\infty(M, S^+; P)$. Since the problem is at the boundary, we can suppose that the support of f is cointained in $N \times [0, 1]$, so that f can be seen as a function on $N \times [0, +\infty]$

and, by reflection, on $N\times\mathbb{R}$. Now, we regularise f by letting act e^{-tB^2} on the boundary direction and by a classical convolution on the cylinder-direction. Hence, $\operatorname{Dom}\mathcal{D}^+ = W^1(M,S^+;P)$ and, similarly, $\operatorname{Dom}\mathcal{D}^- = W^1(M,S^-;1-P)$.

b) Thanks to *a*) and (1.30), we now easily have the inclusion $\operatorname{Dom}(\mathcal{D}^-)^* \supseteq \operatorname{Dom}\mathcal{D}^+ = W^1(M,S^+;P)$, with $(\mathcal{D}^-)^* f = \mathcal{D}^+ f$ for any $f\in W^1(M,S^+;P)$. To conclude, we need to prove $\operatorname{Dom}(\mathcal{D}^-)^* \subseteq W^1(M,S^+;P)$. Let $f\in(\mathcal{D}^-)^*$, so that it exists $g\in L^2(M,S^-)$ such that

$$(f,\mathcal{D}^- h)_{L^2} = (g,h)_{L^2} \quad \text{for all } h\in W^1(M,S^-;1-P) = \operatorname{Dom}\mathcal{D}^-. \tag{1.32}$$

Now, the operator $\mathcal{D}^- R^- - 1 =: S$ has smooth kernel thanks to Remark 1.15. By (1.32) we get, for any $h\in L^2(M,S^+)$,

$$\begin{aligned}(f,h)_{L^2} &= (f,\mathcal{D}^- R^- h)_{L^2} - (f,Sh)_{L^2} = (g,R^- h)_{L^2} - (f,Sh)_{L^2} = \\ &= ((R^-)^* g,h)_{L^2} - (S^* f,h)_{L^2},\end{aligned}$$

that is, $f=(R^-)^* g - S^* f$. Now, S^* is smoothing because S is so, while $(R^-)^*$ takes values in $W^1(M,S^+)$ by construction. In particular, $f\in W^1(M,S^+)$. It remains to prove that f satisfies the boundary condition, i.e $Pf|_N=0$. By separation of variables, for any $h\in W^1(M,S^-)$ we have

$$(f,D^- h)_{L^2} = (\mathcal{D}^+ f,h)_{L^2} + (f|_N,\sigma^{-1}h|_N)_{L^2(N)}.$$

If h satisfies the boundary condition $(1-P)\sigma^{-1}h=0$, by (1.32) we get

$$(\mathcal{D}^+ f,h)_{L^2} + (f|_N,\sigma^{-1}h|_N)_{L^2(N)} = (g,h)_{L^2}.$$

It follows that $(\mathcal{D}^+ f,h)_{L^2} = (g,h)_{L^2}$ for any h with support contained in the interior of M and hence, by density, for any $h\in L^2(M,S^-)$. In particular,

$$(f|_N,\sigma^{-1}h|_N)_{L^2(N)} = 0$$

for any $h\in W^1(M,S^-)$ satisfying $(1-P)\sigma^{-1}h|_N=0$. Since any section in $L^2(N,S^-|_N)$ can be extended to a W^1-section on M, this means that $f|_N\in\ker(1-P)^\perp=\ker P$, hence $f\in W^1(M,S^+;P)$.

□

We now have all the ingredients to prove the following fundamental result.

Proposition 1.17. *$\mathcal{D}^+$ and $\mathcal{D}^-$ (viewed here as bounded operators between Hilbert spaces) are Fredholm operators, with null-spaces made by smooth sections. In particular, $\ker D^+$ and $\ker D^-$ (with boundary conditions) are finite-dimensional and the Fredholm index of $\mathcal{D}^+$ is given by*

$$\operatorname{ind}\mathcal{D}^+ = \dim\ker\mathcal{D}^+ - \dim\ker\mathcal{D}^- = \dim\ker D^+ - \dim\ker D^-. \tag{1.33}$$

Proof. Proposition 1.14 gives us an inverse R^+ of $\mathcal{D}^+$ modulo smoothing operators. Since smoothing operators are compact both as operators $L^2 \to L^2$ and as operators $W^1 \to W^1$, $\mathcal{D}^+$ is a Fredholm operator. Moreover, suppose $\mathcal{D}^+ f = 0$. Then, $f = (R^+\mathcal{D}^+ - K)f = -Kf$. Hence, $f \in C^\infty$ because K has smooth kernel. The same arguments apply to $\mathcal{D}^-$. We have thus proven that $\ker \mathcal{D}^+ = \ker D^+$. Thanks to Proposition 1.16, we also have $(\operatorname{im} \mathcal{D}^+)^\perp = \ker \mathcal{D}^- = \ker D^-$. 1.33 follows then by definition of index. □

We set

$$\operatorname{ind} D^+ := \operatorname{ind} \mathcal{D}^+ \tag{1.34}$$

and we call this integer the *index* of D^+ on the manifold with boundary M. This is the number for which we aim to find a formula.

As a consequence of Proposition 1.16, the second order operators $\mathcal{D}^-\mathcal{D}^+$ and $\mathcal{D}^+\mathcal{D}^-$ are positive and self-adjoint. By functional calculus, for any $t > 0$ we can hence construct the *heat operators* $e^{-t\mathcal{D}^-\mathcal{D}^+}$ and $e^{-t\mathcal{D}^+\mathcal{D}^-}$, which will be smoothing operators on $L^2(M, S^+)$ and $L^2(M, S^-)$ respectively. They will be of course the fundamental solutions for the operators $\frac{\partial}{\partial t} + \mathcal{D}^-\mathcal{D}^+$ and $\frac{\partial}{\partial t} + \mathcal{D}^+\mathcal{D}^-$. We want to generalise McKean-Singer formula (Proposition 0.13) to our setting. In order to do that, we will need the following analogue of Theorem 0.6.

Proposition 1.18. *The eigenvalues μ of $\mathcal{D}^-\mathcal{D}^+$ form a discrete subset of $\mathbb{R}_+$. Each eigenspace $E_\mu \subseteq L^2(M, S^+)$ is finite dimensional and made of smooth sections. We have moreover an orthogonal decomposition*

$$L^2(M, S) = \bigoplus_\mu E_\mu.$$

The operator $\mathcal{D}^+\mathcal{D}^-$ has an analogous decomposition.

Proof. Let H be the domain of $\mathcal{D}^+\mathcal{D}^-$, which is a Hilbert subspace of $W^1(M, S^+)$. The operator $\mathcal{D}^+\mathcal{D}^- : H \to L^2(M, S^+)$ is Fredholm because it admits a parametrix R^+R^-. Let $K := \ker \mathcal{D}^+\mathcal{D}^- \subseteq L^2(M, S^+)$. We have hence an isomorphism of Hilbert spaces

$$\mathcal{D}^+\mathcal{D}^- : H \cap K^\perp \to K^\perp.$$

Let Q be the inverse map. We extend Q to a map on L^2 (the *Green operator* for $\mathcal{D}^+\mathcal{D}^-$) by sending K to 0. Q is hence a bounded operator with values in $W^1(M, S^+)$: by Rellich's lemma, it is compact as an operator on $L^2(M, S^+)$. Moreover, it clearly is self-adjoint. Thanks to the spectral theorem for compact self-adjoint operators, $L^2(M, S^+)$ is a sum of finite-dimensional eigenspaces for Q. The thesis now easily follows. □

Let $\mathcal{E}^+$ and $\mathcal{E}^-$ the smooth kernels of the heat operators $e^{-t\mathcal{D}^-\mathcal{D}^+}$ and $e^{-t\mathcal{D}^+\mathcal{D}^-}$ respectively, called the *heat kernels* on M. The same proof of Proposition 0.13 gives us now our desired heat kernel formula.

Proposition 1.19. *For any $t > 0$, we have*

$$\operatorname{ind} D^+ = \operatorname{tr} e^{-t\mathcal{D}^-\mathcal{D}^+} - \operatorname{tr} e^{-t\mathcal{D}^+\mathcal{D}^-} = \int_M \operatorname{tr}_z(\mathcal{E}^+(t; z, z) - \mathcal{E}^-(t; z, z))dz.$$

As we constructed parametrices for $\mathcal{D}^+$ and $\mathcal{D}^-$ by glueing together parametrices on the cylinder and on the closed double, we want to construct approximate fundamental solutions for $\frac{\partial}{\partial t} + \mathcal{D}^-\mathcal{D}^+$ and $\frac{\partial}{\partial t} + \mathcal{D}^+\mathcal{D}^-$ (that is, approximate heat operators) by glueing together fundamental solutions for the corrisponding operators on the cylinder and on the closed double. Using the functions ϕ_i, ψ_i defined in 1.31, we put, for any $t > 0$,

$$T^+(t) := \phi_1 e^{-t\mathcal{A}^*\mathcal{A}}\psi_1 + \phi_2 e^{-t\widehat{\mathcal{D}}^-\widehat{\mathcal{D}}^+}\psi_2,$$

$$T^-(t) := \phi_1 \sigma e^{-t\mathcal{A}\mathcal{A}^*}\sigma^{-1}\psi_1 + \phi_2 e^{-t\widehat{\mathcal{D}}^+\widehat{\mathcal{D}}^-}\psi_2.$$

For each fixed t, $T^\pm(t)$ is a well defined smoothing operator on $L^2(M, S^\pm)$. The kernels of T^+ and T^- are, respectively,

$$\mathcal{E}_a^+(t; w, z) = \phi_1(w)\mathcal{E}_c(t; w, z)\psi_1(z) + \phi_2(w)\widehat{\mathcal{E}}^+(t; w, z)\psi_2(z) \tag{1.35}$$

and

$$\mathcal{E}_a^-(t; w, z) = \phi_1(w)\sigma\mathcal{E}_{c*}(t; w, z)\sigma^{-1}\psi_1(z) + \phi_2(w)\widehat{\mathcal{E}}^-(t; w, z)\psi_2(z), \tag{1.36}$$

where $\mathcal{E}_c$, $\mathcal{E}_{c*}$ are the heat kernels on the cylinder and $\mathcal{E}^+$, $\widehat{\mathcal{E}}^-$ are the (graded) heat kernels on the double of M. The following result shows that these are truly approximate heat kernels for $t \to 0$.

Proposition 1.20. *There exist constants c, $c' > 0$ such that, for any z, $w \in M$ and small enough t, we have*

$$|\mathcal{E}_a^\pm(t; w, z) - \mathcal{E}^\pm(t; z, w)| \le c\, e^{-c'/t}.$$

Proof. (*Sketch*) From Proposition 0.16 and 1.10, one can show that that $(\frac{\partial}{\partial t} + D^-D^+)T^+$ is exponentially small as $t \to 0$. Moreover, $T^\pm \to 1$ as $t \to 0$.

Let $T_1^+ := (\frac{\partial}{\partial t} + D^-D^+)T^+$, $T_{k+1}^+ := T_k^+ * T_1^+$, where $*$ denotes convolution in t and product of operators. It follows from the preceeding remarks that the true fundamental solution is given by a convergent sum

$$e^{-t\mathcal{D}^-\mathcal{D}^+} = T^+ + \sum_{k\ge1}(-1)^k T_k^+ * T^+,$$

(see [7]). The same arguments apply replacing $+$ with $-$. The assertion follows passing to the (smooth) kernels. □

1.1.3 The Atiyah-Patodi-Singer index formula

We are now ready to prove the index formula that consitutes the main result of this section. First note that, on the double $\widehat{M}$, we also have the ungraded heat operator $e^{-t\widehat{\mathcal{D}}^2} = e^{-t\widehat{\mathcal{D}}^-\widehat{\mathcal{D}}^+} \oplus e^{-t\widehat{\mathcal{D}}^+\widehat{\mathcal{D}}^-}$, with heat kernel $\widehat{\mathcal{E}} = \widehat{\mathcal{E}}^+ \oplus \widehat{\mathcal{E}}^-$. By (0.19), the local supertrace of the ungraded heat kernel on the double of M has an asymptotic expansion as $t \to 0$ of the form

$$\mathrm{str}_z\, \widehat{\mathcal{E}}(t; z, z) \sim \sum_{k\ge -l} \alpha_k(z) t^k, \tag{1.37}$$

where $\alpha_k(z)$ is the local supertrace of the coefficients in the expansion of $\widehat{\mathcal{E}}$ (and $\dim M = 2l$).

Proposition 1.21. *Let the asymptotic expansion of* $\operatorname{str}_z \mathcal{E}(t;z,z)$ *as* $t \to 0$ *be given by* (1.37)*. Then*

$$\operatorname{ind} D^+ = \int_M \alpha_0(z)dz - \frac{h(B)+\eta(B)}{2}, \tag{1.38}$$

where $h(B)$ *is the dimension of the null-space of* B *and* $\eta(B)$ *is defined in Corollary 1.12 by* (1.29)*.*

Proof. 1) The starting point is Proposition 1.19, from which we get

$$\operatorname{ind} D^+ = \int_M \operatorname{tr}_z(\mathcal{E}^+(t;z,z) - \mathcal{E}^-(t;z,z))dz$$

2) Thanks to Proposition 1.20 as $t \to 0$ we can substitute the true heat kernels with the approximate ones, obtaining an asymptotic expansion

$$\sim \int_M \operatorname{tr}_z(\mathcal{E}_a^+(t;z,z) - \mathcal{E}_a^-(t;z,z))dz$$

3) By definition of the approximate heat kernels (formulas (1.35) and (1.36)), since $\phi_i\psi_i = \psi_i$ for $i = 1,2$, we get

$$= \int_0^1 \int_N \operatorname{tr}_{(y,v)}(\mathcal{E}_c(t;y,v,y,v) - \mathcal{E}_{c*}(t;y,v,y,v))\psi_1(v)dydv + \\ + \int_M tr_z(\widehat{\mathcal{E}}^+(t;z,z) - \widehat{\mathcal{E}}^-(t;z,z))\psi_1(z)dz$$

4) Thanks to Remark 1.13, the first integral is exponentially close to our symmetric integral $\mathcal{K}(t) = \int_0^\infty \int_N \operatorname{tr}_{(y,v)}(\mathcal{E}_c(t;y,v,y,v) - \mathcal{E}_{c*}(t;y,v,y,v))dydv$ as $t \to 0$. In the second integral we can suppress ψ_2, because in the collar $N \times [0,1]$ (where ψ_2 fails to be constantly 1) Dirac operators are conjugated; namely, we have $\widehat{\mathcal{D}}^+\widehat{\mathcal{D}}^- = \sigma\widehat{\mathcal{D}}^-\widehat{\mathcal{D}}^+\sigma^{-1}$, hence $\widehat{\mathcal{E}}^- \sim \sigma\widehat{\mathcal{E}}^+\sigma^{-1}$ asymptotically. Therefore, we can write

$$\sim \mathcal{K}(t) + \int_M \operatorname{str}_z \widehat{\mathcal{E}}(t;z,z)dz$$

5) By the asymptotic expansion as $t \to 0$ recalled in the statement, we finally have

$$\operatorname{ind} D^+ \sim \mathcal{K}(t) + \sum_{k \geq -2l} t^{\frac{k}{2}} \int_M \alpha_k(z)dz. \tag{1.39}$$

6) This is our last step. Putting

$$a_k := \begin{cases} \int_M \alpha_k(z)dz & \text{for } k \neq 0 \\ \int_M \alpha_0(z)dz + \operatorname{ind} D^+ & \text{for } k = 0, \end{cases}$$

from (1.39) we get an asymptotic expansion as $t \to 0$

$$\mathcal{K}(t) \sim \sum_{k \geq -l} a_k t^k.$$

From Corollary 1.12 (formula (1.29)) we hence get

$$\eta(B) = -(h(B) + 2a_0) = -\left(h(B) + 2\int_M \alpha_0(z)dz + 2\operatorname{ind} D^+\right).$$

Rearranging the terms, we obtain (1.38).

□

Suppose now that M is oriented. The double $\widehat{M}$ can be given an orientation which extends the one on M, by choosing the opposite orientation on the other half. The local index theorem (Theorem 0.20) then applies to it, so $\alpha_0(z)dz = \alpha \operatorname{vol}$ is the n-form part of

$$\widehat{A}(TM) \operatorname{ch}_s(S/\Phi).$$

From now on, we will denote it by AS and simply call it the *Atiyah-Singer integrand*. The following statement resumes more systematically our main result.

Theorem 1.22. (Atiyah-Patodi-Singer index theorem) *Let M an even-dimensional compact oriented Riemannian manifold with boundary and $S = S^+ \oplus S^-$ a graded Clifford bundle on M, with graded Dirac operator*

$$D : C^\infty(M, S) \to C^\infty(M, S).$$

Suppose that, near the boundary N, M is isometrically diffeomorphic to a product $N \times [0,1]$, so that the positive part of D locally takes the special form

$$D^+ = \sigma\left(\frac{\partial}{\partial u} + B\right),$$

where $\sigma : S^+ \to S^-$ is Clifford multiplication by the inward normal vector and B is a Dirac operator on the closed manifold N. We impose on D^+ the boundary condition

$$P(f|_N) = 0,$$

where $P : L^2(N, S^+|N) \to L^2(N, S^+|N)$ is the spectral projection of B corresponding to the non-negative eigenvalues. The operator

$$D^+ : C^\infty(M, S^+; P) \to C^\infty(M, S^-)$$

has then a finite index given by the formula

$$\operatorname{ind} D^+ = \int_M AS - \frac{h(B) + \eta(B)}{2}, \tag{1.40}$$

where

- $AS := \widehat{A}(TM)\operatorname{ch}_s(S/\Phi)$ *is the Atiyah-Singer integrand of Theorem 0.21;*
- $h(B)$ *is the dimension of the null space of* B*;*
- $\eta(B)$ *is the value at* $s = 0$ *of a meromorfic extension of the function* $\eta(s) = \sum_{\lambda \neq 0} \operatorname{sgn} \lambda |\lambda|^{-s}$*, where* λ *runs over the eigenvalues of* B*.*

1.1.4 Reformulation for manifolds with cylindrical ends

There is another equivalent formulation for the Atiyah-Patodi-Singer theorem, which will be useful for treating the explicit case of the signature operator. Instead of compact manifolds with boundary, we can consider manifolds with an infinite cylindrical end. Any such manifold M_∞ can be obtained by elongating a manifold M with boundary $N = \partial M$ as:

$$M_\infty := (N \times (-\infty, 0]) \cup_N M.$$

If M happens to have a Riemannian structure which is a product $N \times [0,1]$ near the boundary, M_∞ will be a Riemannian manifold with product structure on the whole cylinder $N \times (-\infty, 1]$. A similar remark can be done for the structure of a (graded) Clifford bundle S on M, that will be extended to a Clifford bundle S_∞ on M_∞ in the natural way. Let D_∞ a graded Dirac operator on M_∞. As in Proposition 1.1, in the infinite cylinder $N \times (-\infty, 1]$ it will have the form

$$D^+ = \sigma \left(\frac{\partial}{\partial u} + B \right).$$

Consider now the operators:

$$D^+ : C^\infty(M, S^+; P) \to C^\infty(M, S^-), \qquad D^+_\infty : C^\infty(M_\infty, S^+_\infty) \to C^\infty(M_\infty, S^-_\infty),$$
$$D^- : C^\infty(M, S^-; 1-P) \to C^\infty(M, S^+), \qquad D^-_\infty : C^\infty(M_\infty, S^-_\infty) \to C^\infty(M_\infty, S^+_\infty).$$

We want to find an isomorphism between the null-spaces of the Dirac operators on M (with appropriate boundary conditions) with some subspace of the null-spaces of the corresponding operators on M_∞ (since M_∞ is non-compact, the null-spaces themselves will be in general infinite-dimensional). We will be mainly interested in L^2-sections, but we also need to introduce another space. We denote by $L^2_{\text{ext}}(M_\infty, S_\infty)$ the space of sections f which are locally in L^2 and such that, for large negative u, they can be written in the form

$$f(x,u) = f_{L^2}(x,u) + f_\infty(y), \tag{1.41}$$

where $f_{L^2} \in L^2(M_\infty, S_\infty)$ and $f_\infty \in L^2(N, S|_N)$. We call such an f an *extended L^2-section*, and f_∞ the *limiting value* of f. We then define the following spaces:

$$L^2 \ker D^\pm_\infty := \ker D^\pm_\infty \cap L^2(M_\infty, S^\pm_\infty);$$

$$L^2_{\text{ext}} \ker D^\pm_\infty := \ker D^\pm_\infty \cap L^2_{\text{ext}}(M_\infty, S^\pm_\infty).$$

We are now ready for the following result.

Proposition 1.23. *We have natural isomorphisms:*

(i) $\ker D^+ \cong L^2 \ker D^+_\infty$*;*

(ii) $\ker D^- \cong L^2_{\text{ext}} \ker D^-_\infty$.

Proof. (i) Let $f \in \ker D^+$. Near the boundary, as usual, we can expand f in the form

$$f(x, u) = \sum_\lambda f_\lambda(u)\varphi_\lambda(y).$$

The equation $D^+ f = 0$ translates in $(\partial/\partial u + \lambda) f_\lambda = 0$, i.e. $f_\lambda(u) = e^{-\lambda u} f_\lambda(0)$. The boundary condition implies however that $f_\lambda(0) = 0$ for any $\lambda \geq 0$. Hence, f extends uniquely to a solution of $D^+_\infty f = 0$ on the whole M_∞, which is of the form

$$f(x, u) = \sum_{\lambda<0} e^{-\lambda u} f_\lambda(0)\varphi_\lambda(y),$$

on the infinite cylinder. This clearly belongs to $L^2(M_\infty, S^+_\infty)$.

Conversely, any $f \in L^2 \ker D^+_\infty$ will be of the form

$$f(x, u) = \sum_\lambda e^{-\lambda u} f_\lambda(0)\varphi_\lambda(y).$$

All the $f_\lambda(0)$ must hence be 0 for $\lambda \geq 0$, otherwise f would not be in L^2. As a consequence, restriction of f to M respects the boundary condition and hence is in $\ker D^+$.

(ii) Let $g \in \ker D^-$. Here we can write

$$g(x, u) = \sum_\lambda g_\lambda(u)(\sigma\varphi_\lambda)(y).$$

The equation $D^- g = 0$ translates now into $(-\partial/\partial u + \lambda) g_\lambda = 0$, i.e. $g_\lambda(u) = e^{\lambda u} g_\lambda(0)$. The boundary condition is now $g_\lambda(0) = 0$ for $\lambda < 0$. Hence

$$g(x, u) = \sum_{\lambda\geq 0} e^{\lambda u} g_\lambda(0)(\sigma\varphi_\lambda)(y).$$

This section extends uniquely to an extended L^2-solution of $D^-_\infty g = 0$ on M_∞, with limiting value $g_\infty(y) = g_0(0)(\sigma\varphi_0)(y)$ (this is in fact a finite sum over the eigenvalues $\lambda = 0$).

Conversely, any $g \in L^2_{\text{ext}} \ker D^-_\infty$ will be of the form

$$g(x,u) = \sum_\lambda e^{\lambda u} g_\lambda(0)(\sigma\varphi_\lambda)(y),$$

and such a formula can only give an extended L^2 section (with limiting value given at $\lambda = 0$) if $g_\lambda(0) = 0$ for any $\lambda < 0$. Hence, its restriction to M respects the boundary condition.

□

As a first consequence, the L^2-null-spaces are both finite-dimensional. Moreover, we have

$$\operatorname{ind} D^+ = \dim L^2 \ker D^+_\infty - \dim L^2 \ker D^-_\infty - h^-, \tag{1.42}$$

where h^- denotes the dimension of the space of sections annihilated by $B\sigma^{-1}$ which are limiting values to sections in $L^2_{\text{ext}} \ker D^-_\infty$ (we do not use the term "index" for the number $\dim L^2 \ker D^+_\infty - \dim L^2 \ker D^-_\infty$ because it is not a real Fredholm index). The Atiyah-Patodi-Singer index formula (1.40) can be thus rewritten in this context in the following manner.

Corollary 1.24.

$$\dim L^2 \ker D^+_\infty - \dim L^2 \ker D^-_\infty = \int_M AS + h^- - \frac{1}{2}h(B) - \frac{1}{2}\eta(B). \tag{1.43}$$

As we defined h^-, we can simmetrically define h^+ to be the dimension of the subspace of $\ker B$ which are limiting values to sections in $L^2_{\text{ext}} \ker D^+_\infty$. A remarkable fact about these two spaces of limiting values is that their dimensions add up to the full dimension of $\ker B$. In other words:

Proposition 1.25. $h(B) = h^+ + h^-$.

Proof. If from the beginning of the section we choose D^- instead of D^+, with boundary condition correspondig to the spectral projection for $\lambda \leq 0$, we obtain the following index formula:

$$\operatorname{ind} D^- = -\int AS - \frac{h(B) - \eta(B)}{2}. \tag{1.44}$$

In fact, to get a desired expression near the boundary of the form $D^- = \sigma^{-1}(\partial/\partial u + B')$, we have to choose $B' = -\sigma B \sigma^{-1}$ which compared to B has isomorphic null-space and opposite eigenvalues. The integrand also changes sign as it comes from a difference of heat kernels.

The equivalent of (1.42) is of course

$$\operatorname{ind} D^- = \dim L^2 \ker D^-_\infty - \dim L^2 \ker D^+_\infty - h^+$$

Thanks to (1.44), we get hence the following analogue of (1.43):

$$\dim L^2 \ker D^-_\infty - \dim L^2 \ker D^+_\infty = -\int_M AS + h^+ - \frac{1}{2}h(B) + \frac{1}{2}\eta(B). \tag{1.45}$$

Summing (1.43) with (1.45), we get

$$0 = h^- + h^+ - h(B),$$

from which the thesis follows. ☐

Corollary 1.26. *The index formula* (1.43) *can once more be rewritten as*

$$\dim L^2 \ker D^+_\infty - \dim L^2 \ker D^-_\infty = \int_M AS + \frac{1}{2}(h^+ - h^- - \eta(B)). \tag{1.46}$$

We conclude the section with one more result that will be useful in the following.

Proposition 1.27. *The extended L^2-solutions of D^2_∞ and D_∞ coincide on M_∞. In particular, the true L^2-solutions also coincide for these operators.*

Proof. It will suffice to prove that, for any extended L^2 section $f \in C^\infty(M_\infty, S_\infty)$, $D^2_\infty f$ implies $D_\infty f = 0$. We will work in the positive part S^+_∞: the proof for the negative part is analogous.

Let hence f an extended L^2-solution of $D^-_\infty D^+_\infty f = 0$. Expanding f in the cylinder as $f = \sum f_\lambda \varphi_\lambda$, $D^-_\infty D^+_\infty f = 0$ yelds

$$\left(-\frac{\partial^2}{\partial u} + \lambda^2\right) f_\lambda = 0 \qquad \text{for any } \lambda.$$

Resolving this family of ordinary differential equations, we see that f is of the form

$$f(x,u) = \sum_\lambda (a_\lambda e^{\lambda u} + b_\lambda e^{-\lambda u})\varphi_\lambda(x).$$

Since f is an extended L^2-section, we must have $a_\lambda = 0$ for $\lambda < 0$ and $b_\lambda = 0$ for $\lambda > 0$. Hence,

$$f(x,u) = \sum_{\lambda \geq 0} a_\lambda e^{\lambda u}\varphi_\lambda(x) + \sum_{\lambda \leq 0} b_\lambda e^{-\lambda u}\varphi_\lambda(x).$$

Let us show that the L^2 norm of $f(\cdot, u)$ on $N \times \{u\}$ is bounded as $u \to -\infty$:

$$\begin{aligned} \|f(\cdot,u)\|^2_{L^2} &= \sum_{\lambda>0} |a_\lambda|^2 e^{2\lambda u} + \sum_{\lambda<0} |b_\lambda|^2 e^{-2\lambda u} + |a_0 + b_0|^2 \leq \\ &\leq \sum_{\lambda>0} |a_\lambda|^2 + \sum_{\lambda<0} |b_\lambda|^2 + |a_0 + b_0|^2 = \|f(\cdot,0)\|^2_{L^2} < \infty. \end{aligned} \tag{1.47}$$

The L^2 norm of $D^+ f(\cdot, u)$ must be even exponentially decaying, in fact

$$D^+ f(x,u) = 2\sum_{\lambda>0} \lambda a_\lambda e^{\lambda u} \cdot (\sigma\varphi_\lambda)(x)$$

and hence

$$\|D_\infty^+ f(\cdot,u)\|_{L^2}^2 = 4\sum_{\lambda>0}\lambda^2|a_\lambda|^2 e^{2\lambda u} \leq \left(\sum_{\lambda>0}\lambda^2|a_\lambda|^2\right) e^{\alpha u} = Ce^{\alpha u} \tag{1.48}$$

(it suffices to take α such that $\alpha < 2\lambda$ for any $\lambda > 0$). Denote now by M_u the compact manifold with boundary $(N \times [u,0]) \cup_N M \subseteq M_\infty$. We call N_u its boundary, which is a copy of N. Recalling now that D_∞^+ and D_∞^- are formally adjoint, integrating by parts we get

$$\int_{M_u}(D_\infty^- D_\infty^+ f(z), f(z))dz =$$

$$= \int_{M_u}(D_\infty^+ f(z), D_\infty^+ f(z))dz + \int_{N_u}(D_\infty^+ f(x,u), \sigma f(x,u))dx.$$

By hypothesis, the first term is 0. Moreover, thanks to (1.47) and (1.48), by Cauchy-Schwarz inequality, the boundary integral tends to 0 as $u \to -\infty$. As a consequence, we get

$$\int_{M_\infty}(D_\infty^+ f(z), D_\infty^+ f(z)) = \lim_{u\to 0}\int_{M_u}(D_\infty^+ f(z), D_\infty^+ f(z))dz = 0,$$

or, equivalently $D_\infty^+ f = 0$. □

1.2 The signature theorem for manifolds with boundary

Suppose now that M^{2l} is compact oriented manifold of even dimension. As in the previous chapter, we can define a structure of Clifford bundle on the complexified exterior bundle $\Lambda^* := \Lambda^*(T^*M) \otimes \mathbb{C}$, whose Dirac operator is $D = d+\delta$. A grading on $\Lambda^* = \Lambda^+ \oplus \Lambda^-$ can be obtained by the ± 1-eigenbundles of the grading operator $\tau = i^{p(p-1)+l}\star$ on p-forms. Provided with this grading, D is called the signature operator on M.

If M is without boundary, the index of D^+ equals the signature of M (Proposition 0.13). In the case where $M = M^{4k}$ is a manifold with boundary, the signature can also be defined. We will study how this number is related to the index of D^+ with Atiyah-Patodi-Singer boundary condition. Thanks to the index theorems of the previous section, this will permit us to give a generalisation of the Hirzebruch signature theorem (Theorem 0.23) valid in the case of compact manifolds with boundary. Namely, we shall prove the formula

$$\operatorname{sign} M = \int_M L(M) - \frac{1}{2}\eta(B),$$

where B is the so-called *odd signature operator*, acting on the space of complex differential forms on ∂M.

The above formula is particularly remarkable, as it relates three invariants coming from different areas of mathematics. In fact, the signature of M is a topological invariant of the manifold, the Hirzebruch L-polynomial comes from the differential geometry of M, and η is a spectral invariant.

1.2.1 Hodge theory on manifolds with boundary

We recall here Hodge decomposition theorems for complete manifolds with and without boundary. Proofs can be found in [11] and [22] for the classical version and the boundary version respectively.

First, we present the classical version, for manifolds without boundary.

Theorem 1.28. (Hodge-Kodaira decomposition) *Let X be a complete oriented Riemannian manifold without boundary. Then, for each p, we have an orthonormal decomposition*

$$L^2\Omega^p(X) = \mathcal{H}^p_{(2)}(X) \oplus \overline{d(\Omega_c^{p-1}(X))} \oplus \overline{\delta(\Omega_c^{p+1}(X))}, \tag{1.49}$$

where $\mathcal{H}^*_{(2)}(X) = L^2 \ker \Delta$ is the space of L^2-harmonic forms on M.

In the context of complete manifolds with boundary, one has to deal with boundary conditions. In order to achieve a good level of generality, it is convenient to suppose that the boundary $Y := \partial X$ is written as the disjoint union $Y = Y_1 \sqcup Y_2$ of possibly empty manifolds. We impose then *relative boundary conditions* on Y_1 and *absolute boundary conditions* on Y_2, i.e. we define

$$\begin{aligned} \Omega^*_d(X, Y_1) &:= \{\omega \in \Omega^*_c(X) \,|\, \omega|_{Y_1} = 0\}, \\ \Omega^*_\delta(X, Y_1) &:= \{\omega \in \Omega^*_c(X) \,|\, (\star\omega)|_{Y_2} = 0\}. \end{aligned}$$

We then define the space of L^2-harmonic forms satisfying absolute/relative boundary conditions by

$$\mathcal{H}^*_{(2)}(X, Y_1) := \{\omega \in L^2 \ker \Delta \,|\, \delta\omega|_{Y_1} = 0, \star d\omega|_{Y_2} = 0, \omega|_{Y_1} = 0, (\star\omega)|_{Y_2} = 0\}. \tag{1.50}$$

The correct generalisation of Theorem 1.28 is the following.

Theorem 1.29. *Let X be a complete oriented Riemannian manifold with boundary $Y = Y_1 \sqcup Y_2$. Then, for each p, we have*

$$L^2\Omega^p(X) = \mathcal{H}^p_{(2)}(X, Y_1) \oplus \overline{d(\Omega_d^{p-1}(X, Y_1))} \oplus \overline{\delta(\Omega_\delta^{p+1}(X, Y_1))}.$$

Let us now introduce the following notations:

$$\begin{aligned} \mathcal{H}^*_{(2)}(X, Y)_{\text{abs}} &:= \{\omega \in L^2 \ker \Delta \,|\, \star d\omega|_Y = 0, (\star\omega)|_Y = 0\}, \\ \mathcal{H}^*_{(2)}(X, Y)_{\text{rel}} &:= \{\omega \in L^2 \ker \Delta \,|\, \delta\omega|_Y = 0, \omega|_Y = 0\}. \end{aligned}$$

Setting $Y_1 = \emptyset$ or $Y_2 = \emptyset$, Theorem 1.29 can be specialised into the two following decompositions.

Corollary 1.30.

$$L^2\Omega^p(X) = \mathcal{H}^p_{(2)}(X, \partial X)_{\mathrm{abs}} \oplus \overline{d(\Omega_c^{p-1}(X)} \oplus \overline{\delta(\{\omega \in \Omega_c^{p+1}(X) \mid \star\omega|_{\partial X} = 0\}},\tag{1.51}$$

$$L^2\Omega^p(X) = \mathcal{H}^p_{(2)}(X, \partial X)_{\mathrm{rel}} \oplus \overline{d(\Omega_c^{p-1}(X, Y))} \oplus \overline{\delta(\Omega_c^{p+1}(X))}.\tag{1.52}$$

1.2.2 The odd signature operator

Let here $M = M^{2l}$, and $N = N^{2l-1}$ be its boundary. Suppose that M is isometrically diffeomorphic to a product $N \times [0, 1]$ near the boundary, as in the previous section. The positive part of the Dirac operator has hence, as usual, the local special form

$$D^+ = \sigma\left(\frac{\partial}{\partial u} + B\right),$$

where $B : C^\infty(N, \Lambda^+|_N) \to C^\infty(N, \Lambda^+|_N)$. The bundle $\Lambda^+|_N$ can be identified with $\Lambda^*(T^*N) \otimes \mathbb{C}$ by the isomorphism

$$\begin{aligned} \Lambda^*(T^*N)\otimes\mathbb{C} &\to \Lambda^+|_N \\ \omega &\mapsto \omega + \tau\omega. \end{aligned}$$

Using this identification, B can be viewed as an operator

$$B : \Omega^*(N) \to \Omega^*(N).$$

By formula (1.3) and the explicit definition of the Clifford action, one can compute B to act as

$$B = i^{p(p+1)+l}(d\star + (-1)^{p+1} \star d) \qquad \text{on } \Omega^p(N),$$

where $\star$ is here the Hodge star operator on N. B is called the *odd signature operator* on N.

Remark 1.31. *Notice that B equals the Dirac operator $d + \delta$ on N modulo a bundle isomorphism. In particular, thanks to the Hodge theorem (Theorem 0.12), the map*

$$\begin{aligned} \ker B &\to H^*(N) \\ \omega &\mapsto [\omega] \end{aligned}$$

is a vector space isomorphism.

1.2.3 The signature of a manifold with boundary

Suppose now that the dimension of M is divisible by 4, so that $M = M^{4k}$. We want to generalise the notion of the signature of M, given in 0.1.4 for closed manifolds, to the case where M is compact with boundary.

If N is the boundary of M, we can consider the relative cohomology $H^*(M,N)$, coming from the complex $\Omega^*(M,N)$ of differential forms whose pullback to the boundary is 0. We have hence a well defined Hermitian form

$$s : H^{2k}(M,N) \times H^{2k}(M,N) \to \mathbb{C}$$
$$([\alpha],[\beta]) \to \int_M \alpha \wedge \overline{\beta}.$$

Contrarily to the case where $N = \emptyset$, this form may now be degenerate. Nonetheless, we can still define the *signature* of M as

$$\operatorname{sign} M := \operatorname{sign} s$$

(that is, positivity minus negativity index).

Consider now the natural map

$$\varphi : H^*(M,N) \to H^*(M),$$

and call φ^p its restriction to H^p. We set

$$\widehat{H}^*(M) := \operatorname{im} \varphi. \tag{1.53}$$

We have the following result.

Proposition 1.32. *Let* $\operatorname{rad} s \subseteq H^{2k}(M,N)$ *denote the radical of* s*, that is, the space of elements* a *such that* $s(a,b) = 0$ *for all* b*. Then*

$$\ker(\varphi^{2k}) \subseteq \operatorname{rad} s$$

(in fact, they can be shown to coincide). In particular, s *descends to a Hermitian form*

$$\widehat{s} : \widehat{H}^{2k}(M) \times \widehat{H}^{2k}(M) \to \mathbb{C}$$

such that

$$\operatorname{sign} \widehat{s} = \operatorname{sign} s = \operatorname{sign} M.$$

Proof. Let $[\alpha]_\partial \in \ker(\varphi^{2k})$. Then, there exists $\gamma \in \Omega^{2k-1}(M)$ such that $d\gamma = \alpha$. As a consequence, for each $[\beta] \in H^{2k}(M,N)$ (so that β is closed and pullbacks to 0 on N) we have

$$s([\alpha]_\partial,[\beta]_\partial) = \int_M \alpha \wedge \overline{\beta} = \int_M d\gamma \wedge \overline{\beta} = \int_M d(\gamma \wedge \overline{\beta}) + \int_M \gamma \wedge d\overline{\beta} =$$
$$= \int_M d(\gamma \wedge \overline{\beta}) + 0 = \int_N \gamma|_N \wedge \overline{\beta}|_N = 0.$$

Hence, $[\alpha]_\partial \in \operatorname{rad} s$ and the assertion follows. □

Remark 1.33. *Notice that the induced form* $\widehat{s}$ *is not defined by* $([\alpha],[\beta]) \mapsto \int_M \alpha \wedge \overline{\beta}$ *for any* α *and* β*, since this integral may depend on the representative. To compute* $\widehat{s}$*, we have to actually choose representatives* α *and* β *whose pullback to the boundary is* 0*.*

In order to compare the signature of M with the index of the signature operator, as in the closed case we will need some kind of Hodge theory. Using the product structure near the boundary, we can define M_∞ to be the cylindrical elongation of M. Recall that, thanks to Proposition 1.27, the L^2-null-spaces of $D_\infty := d + \delta$ and $D_\infty^2 := \Delta_\infty$ coincide on M_∞. We set

$$\mathcal{H}^*_{(2)}(M_\infty) := L^2 \ker D_\infty = L^2 \ker \Delta_\infty$$

and call its elements the L^2-harmonic forms. On $\mathcal{H}^*_{(2)}(M_\infty)$, we can define the intersection form

$$\begin{aligned} s_\infty : \mathcal{H}^*_{(2)}(M_\infty) \times \mathcal{H}^*_{(2)}(M_\infty) &\to \mathbb{C} \\ (\alpha, \beta) &\to \int_{M_\infty} \alpha \wedge \overline{\beta}. \end{aligned}$$

In fact, s_∞ is equivalent to the sesquilinear form $\widehat{s}$ defined above.

Proposition 1.34. *Let $\widehat{H}^*(M)$ as in (1.53). The map*

$$\begin{aligned} \Phi : \mathcal{H}^*_{(2)}(M_\infty) &\to \widehat{H}^*(M) \\ \omega &\mapsto [\omega|_M] \end{aligned}$$

is well defined, and it is an isomophism of graded complex vector spaces. Moreover, it transforms the sesquilinear form s_∞ into $\widehat{s}$, so that

$$\operatorname{sign} M = \operatorname{sign} \widehat{s} = \operatorname{sign} s_\infty.$$

Proof. First of all, notice that the element $[\omega|_M] \in H^*(M)$ is well defined whenever ω is an L^2-harmonic form, since as usual $(d+\delta)\omega = 0$ implies $d\omega = 0$. We want to show that this class is in the image of $H^*(M, N) \to H^*(M)$, which by exactness equals the kernel of $H^*(M) \to H^*(N)$; it suffices hence to show that $\omega|_N$ is exact on N. To see this, we will prove that $\int_\gamma \omega = 0$ for each smooth cycle $\gamma \subseteq N$. Since ω is closed, by Stokes theorem on $\gamma \times [u, 0]$ we see that

$$\int_\gamma \omega = \int_{\gamma_u} \omega$$

for each $u < 0$ (where $\gamma_u := \gamma \times \{u\}$). Since ω is exponentially decreasing as $u \to -\infty$ (see the proof of 1.23), we necessarily have $\int_\gamma \omega = 0$. We have thus prooved that Φ takes values in $\widehat{H}^*(M)$.

To show surjectivity, we use the Hodge-Kodaira decomposition of Theorem 1.28. If $[\psi] \in \widehat{H}^*(M)$, by definition we can take ψ to be 0 on N. Extend it by 0 to an L^2-form ψ_∞ on the whole M_∞. Integration by parts shows that ψ_∞ is orthogonal to $\delta(\Omega^*_c(M_\infty)$. By (1.49), hence, we can write it as

$$\psi_\infty = h + \theta,$$

with h harmonic and $\theta \in \overline{d(\Omega_c^*(M_\infty))}$. Restriction to M now gives

$$\psi = h|_M + \theta|_M,$$

with $\theta|_M \in \overline{d(\Omega_c^*(M_\infty))} = d(\Omega_c^*(M_\infty))$. We have then $[\psi] = [\omega|_N] = \Phi(\omega)$.

We now prove that Φ transforms the sesquilinear form s_∞ into $\widehat{s}$. Injectivity will follow for the $2k$-forms, since s_∞ is clearly non degenerate (for any $\omega \neq 0$, $\int_{M_\infty} \omega \wedge \overline{\omega} > 0$). For the other grades, it suffices to replace the above sesquilinear forms by the corresponding ones on $\mathcal{H}^p_{(2)}(M_\infty) \times \mathcal{H}^{4k-p}_{(2)}(M_\infty)$ and $\widehat{H}^p(M) \times \widehat{H}^{4k-p}(M)$.

Let hence h and h' two L^2-harmonic forms on M_∞. We want to show that $s_\infty(h,h') = \widehat{s}([h|_M],[h'|_M])$. We recall that $s_\infty(h,h')$ is defined by

$$s_\infty(h,h') = \int_{M_\infty} h \wedge \overline{h'},$$

while $\widehat{s}([h|_M],[h'|_M])$ is given by

$$\widehat{s}([h|_M],[h'|_M]) = \int_M \omega \wedge \overline{\omega'},$$

where ω, ω' are cohomologous to $h|_M, h'|_M$ respectively and pullback to 0 on the boundary N. Let $\beta, \beta' \in \Omega^{2k-1}(M)$ be such that

$$h|_M = \omega + d\beta, \qquad h'|_M = \omega' + d\beta'.$$

We define moreover $\alpha := \beta|_N, \alpha' := \beta'|_N$, so that

$$h|_N = d\beta|_N = d\alpha, \qquad h'|_N = d\beta'|_N = d\alpha'.$$

We want to show that $\int_{M_\infty} h \wedge \overline{h'} = \int_M \omega \wedge \overline{\omega'}$. Integrating by parts, we find

$$\int_M \omega \wedge \overline{\omega'} = \int_M (h - d\beta) \wedge \overline{(h' - d\beta')} = \int_M h \wedge \overline{h'} - \int_N \alpha \wedge \overline{d\alpha'}.$$

We have reduced ourselves to prove that $\int_{N\times(-\infty,0]} h \wedge \overline{h'} = -\int_N \alpha \wedge \overline{d\alpha'}$. We shall use a Hodge decomposition on the complete manifold with boundary $N \times (-\infty, 0)$. Mirroring the above definitions, we choose closed forms $\widetilde{\omega}, \widetilde{\omega}'$ on $N \times (-\infty, 0]$ which are cohomologous to (the restrictions of) h, h' respectively and pullabck to 0 on $N = N \times \{0\}$. We set $\widetilde{\beta}, \widetilde{\beta}'$ such that

$$h|_{N\times(-\infty,0]} = \widetilde{\omega} + d\widetilde{\beta}, \qquad h'|_{N\times(-\infty,0]} = \widetilde{\omega}' + d\widetilde{\beta}'.$$

By the decomposition (1.52), we can write

$$\widetilde{\omega} = k + \psi, \qquad \widetilde{\omega}' = k' + \psi',$$

where k, k' are harmonic and restrict to 0 on the boundary, while $\psi, \psi' \in \overline{d(\Omega_c^{2k-1}(N \times (-\infty, 0])}$. Thanks to this last property, integration by parts easily shows that

$$\int_{N\times(-\infty,0]} \widetilde{\omega} \wedge \overline{\widetilde{\omega}'} = \int_{N\times(-\infty,0]} k \wedge \overline{k'}.$$

By the product structure, we can now write $k = k_1(t) + k_2(t)dt$, $k' = k'_1(t) + k'_2(t)dt$, with $k_i(t), k'_i(t)$ harmonic forms which only take values in the N-direction. The boundary conditions imply $k_1(0) = 0 = k'_1(0)$. Thus, by harmonicity, we have $k_1(t) = 0 = k'_1(t)$ for any t. As a consequence,

$$k \wedge \overline{k'} = k_2(t)\overline{k'_2(t)}dt \wedge dt = 0.$$

We can now conclude by writing

$$\begin{aligned} 0 = \int_{M\times(-\infty,0]} k \wedge \overline{k'} &= \int_{N\times(-\infty,0]} \widetilde{\omega} \wedge \overline{\widetilde{\omega}'} = \int_{N\times(-\infty,0]} (h - d\widetilde{\beta}) \wedge \overline{(h' - d\widetilde{\beta}')} = \\ &= \int_{N\times(-\infty,0]} h \wedge \overline{h'} - \int_{-N} \alpha \wedge \overline{d\alpha'} = \int_{N\times(-\infty,0]} h \wedge \overline{h'} + \int_{N} \alpha \wedge \overline{d\alpha'}, \end{aligned}$$

where once again we have integrated by parts to show the fourth equality. $\square$

1.2.4 The Atiyah-Patodi-Singer signature formula

Thanks to Proposition 1.34, we can relate the signature of M with index theory.

Proposition 1.35. *Let M^{4k} an oriented Riemannian manifold with boundary of dimention $4k$, with product structure near the boundary, and let $D = d + \delta$ the signature operator on M. Then*

$$\operatorname{sign} M = \dim L^2 \ker D_\infty^- - \dim L^2 \ker D_\infty^+,$$

where D_∞ is the signature operator on the noncompact elongation M_∞ of M.

Proof. By definition, $\dim L^2 \ker D_\infty^- - \dim L^2 \ker D_\infty^+ = \dim \mathcal{H}_{(2)}^+(M_\infty) - \dim \mathcal{H}_{(2)}^+(M_\infty)$. Thanks to Proposition 1.34, the signature can be defined from the nondegenerate Hermitian form

$$\begin{aligned} \mathcal{H}_{(2)}^*(M_\infty) \times \mathcal{H}_{(2)}^*(M_\infty) &\to \mathbb{C} \\ (\alpha, \beta) &\mapsto \int_{M_\infty} \alpha \wedge \overline{\beta}. \end{aligned}$$

From this point, the proof can be carried in the extact same way as the one of Proposition 0.13. $\square$

Combining Proposition 1.35 with our index formula (1.46), we get

$$\operatorname{sign} M = \int_M L(M) + \frac{1}{2}(h^+ - h^- - \eta(B)) \tag{1.54}$$

(Proposition 0.22 applies to the Atiyah-Singer integrand, so that $AS = L(M)$). We recall that h^+ and h^- are dimensions of the spaces of limiting values of *extended L^2-harmonic forms*. In fact, for topological reasons, the following equality holds.

Proposition 1.36. $h^+ = h^-$.

Proof. Let $\mathcal{K}^*(M_\infty) := L^2_{\text{ext}} \ker D_\infty = L^2_{\text{ext}} \ker \Delta_\infty$ be the space of extended L^2-harmonic forms (the second equality is true thanks to Proposition 1.27), and $\mathcal{K}^\pm(M_\infty)$ its graded parts. An element ω of $\mathcal{K}^\pm(M_\infty)$ is of the form

$$\omega = \omega_{L^2} + \omega_\infty \pm \tau\omega_\infty,$$

with $\omega_{L^2} \in \mathcal{H}^*_{(2)}(M_\infty)$ and $\omega_\infty \in \ker B \subseteq \Omega^*(N)$. The number $h^\pm(B)$ by definition is equal to the rank of

$$\begin{aligned} \Pi^\pm : \mathcal{K}^\pm(M_\infty) &\to \ker B \\ \omega &\mapsto \omega_\infty. \end{aligned}$$

This map can be factorised as

$$\mathcal{K}^\pm(M_\infty) \to H^*(M) \xrightarrow{j^*} H^*(N) \xrightarrow{\sim} \ker B,$$

where the first map is $\omega \mapsto [\omega|_M]$, the second is the natural map and the third is the inverse of the one in Remark 1.31. In particular,

$$h^\pm(B) = \dim \operatorname{im} \Pi^\pm \leq \dim \operatorname{im} j^*. \tag{1.55}$$

We can now write the following commutative diagram, coming from Lefschetz duality applied to the exact sequences in cohomology and homology ($n = 4k$ is the dimension of M):

$$\begin{array}{ccccccccc}
\longrightarrow & H^s(M) & \xrightarrow{j^*} & H^s(N) & \xrightarrow{\delta^*} & H^{s+1}(M,N) & \xrightarrow{p^*} & H^{s+1}(M) & \longrightarrow \\
& \downarrow \wr & & & & \downarrow \wr & & \downarrow \wr & \\
\longrightarrow & H_{n-s}(M,N) & \xrightarrow{\delta_*} & H_{n-s-1}(N) & \xrightarrow{j_*} & H_{n-s-1}(M) & \xrightarrow{p_*} & H_{n-s-1}(M,N) & \longrightarrow .
\end{array}$$

From commutativity and exactness of the rows, we easily get

$$\begin{aligned} \dim H^*(N) &= \dim \ker \delta^* + \dim \operatorname{im} \delta^* = \dim \operatorname{im} j^* + \dim \ker p^* = \\ &= \dim \operatorname{im} j^* + \dim \ker p_* = \dim \operatorname{im} j^* + \dim \operatorname{im} j_*. \end{aligned}$$

It is clear however that

$$\dim \operatorname{im}\{j_* : H_*(N) \to H_*(M)\} = \dim \operatorname{im}\{j^* : H^*(M) \to H^*(N)\},$$

since $H^*(\cdot) \cong \operatorname{Hom}(H_*(\cdot), \mathbb{C})$ and the $\operatorname{Hom}(\cdot, \mathbb{C})$ functor is exact. We have then

$$\dim \operatorname{im} j^* = \frac{1}{2} \dim H^*(N) = \frac{1}{2} h(B).$$

Putting this into (1.55), we get

$$h^{\pm}(B) \leq \frac{1}{2} h(B).$$

Thanks to Proposition 1.25, we must have $h^{+} + h^{-} = h(B)$. By our inequality, this forces

$$h^{+} = h^{-} = \frac{1}{2} h(B).$$

□

Thanks to Proposition 1.36, a cancellation in formula (1.54) occurs. As a consequence, we reach the final form of our signature formula. We summarise the result in the next theorem, in which we recall the needed hypothesis.

Theorem 1.37. *Let M^{4k} an oriented Riemannian manifold with boundary of dimention $4k$, with product structure near the boundary N, and let $D = d + \delta$ the signature operator on M. Then*

$$\operatorname{sign} M = \int_M L(M) - \frac{1}{2}\eta(B), \tag{1.56}$$

where

- *$L(M)$ is the Hirzebruch L-polynomial in the Pontryagin classes of the Levi-Civita connection of M;*
- *$B : \Omega^*(N) \to \Omega^*(N)$ is the odd signature operator on N, acting as $i^{p(p+1)+l}(d\star + (-1)^{p+1} \star d)$ on Ω^p;*
- *$\eta(B)$ is the value at $s = 0$ of the eta function $\eta(s) = \sum_{\lambda \neq 0} \operatorname{sgn} \lambda |\lambda|^{-s}$, where λ runs over the eigenvalues of B.*

Remark 1.38. *The odd signature operator B preserves the parity of forms, so we can split it as a direct sum $B^{\mathrm{ev}} \oplus B^{\mathrm{odd}}$. We have in fact a commutative diagram*

$$\begin{array}{ccc} \Omega^{\mathrm{ev}}(N) & \xrightarrow{B^{\mathrm{ev}}} & \Omega^{\mathrm{ev}}(N) \quad , \\ \downarrow \wr & & \downarrow \wr \\ \Omega^{\mathrm{odd}}(N) & \xrightarrow{B^{\mathrm{odd}}} & \Omega^{\mathrm{odd}}(N). \end{array}$$

where the vertical arrows are represented by the bundle isomorphism given by $i^{p(p+1)+l}\star$ on p-forms. B^{ev} and B^{odd} have hence the same eigenvalues, and then

$$\eta(B) = \eta(B^{\mathrm{ev}}) + \eta(B^{\mathrm{odd}}) = 2\eta(B^{\mathrm{ev}}).$$

In particular, we can rewrite (1.56) as

$$\operatorname{sign} M = \int_M L(M) - \eta(B^{\mathrm{ev}}). \tag{1.57}$$

As a consequence of the signature theorem for manifolds with boundary, we will deduce the so-called *Novikov additivity* for the signature, which may also be proved by topological methods.

Corollary 1.39. (Novikov additivity) *Let M and M' two compact oriented manifolds with boundary of dimension $4k$, and suppose that $N \subseteq \partial M$, $-N \subseteq M'$ are two unions of boundary components which are related by an orientation-reversing diffeomorphism. Then, the signature of the oriented manifold $M \cup_N M'$ is computed by*

$$\operatorname{sign}(M \cup_N M') = \operatorname{sign} M + \operatorname{sign} M'.$$

Proof. Choose two metrics for M and M' with product structure near the boundary, and such that N and N' are isometric. Then, the manifold $M \cup_N M'$ inherits a Riemannian metric too. It suffices now to apply the signature formula for these three manifolds, and to notice that the eta invariant is additive for disjoint union and changes sign for reversed orientation. □

1.3 More on the eta invariant

In Section 1.1, we have defined the eta function as

$$\eta(s) = \sum_{\lambda \neq 0} \operatorname{sgn} \lambda \, |\lambda|^{-s}, \tag{1.7}$$

where λ runs over the eigenvalues of the Dirac operator B, defined on a closed odd-dimensional Riemannian manifold N.

The definition of η only depends on structure defined on N, but to show finiteness at $s = 0$ we have appealed to the fact that our N appeared as the boundary of an even-dimensional manifold M, and B as a boundary operator. In general, this is not verified. We now want to show that, for compatible Dirac operators, finiteness at 0 is automatically verified, so that the eta invariant $\eta(B)$ can be defined on any odd dimensional manifold.

1.3.1 A different formula for the eta function

Let B a Dirac operator on a closed Riemannian manifold N^n (here, not necessarily of odd dimension), and let η the associated eta function, defined where the series in (1.7) converges absolutely. Formally, we can write

$$\begin{aligned}
\eta(s) &= \sum_{\lambda\neq 0} \operatorname{sgn}\lambda\,|\lambda|^{-s} = \sum_{\lambda\neq 0}\lambda\,(\lambda^2)^{-\frac{s+1}{2}} = \\
&= \frac{1}{\Gamma(\frac{s+1}{2})}\sum_{\lambda\neq 0}\lambda\,(\lambda^2)^{-\frac{s+1}{2}}\int_0^{+\infty} r^{\frac{s+1}{2}-1}e^{-r}dr = \\
&= \frac{1}{\Gamma(\frac{s+1}{2})}\sum_{\lambda}\lambda\int_0^{+\infty} t^{\frac{s+1}{2}-1}e^{-t\lambda^2}dt = \qquad (1.58)\\
&= \frac{1}{\Gamma(\frac{s+1}{2})}\int_0^{+\infty} t^{\frac{s+1}{2}-1}\left(\sum_{\lambda}\lambda e^{-t\lambda^2}\right)dt = \\
&= \frac{1}{\Gamma(\frac{s+1}{2})}\int_0^{+\infty} t^{\frac{s+1}{2}-1}\operatorname{tr}(Be^{-tB^2})dt,
\end{aligned}$$

where $\Gamma(r) := \int_0^{+\infty}\xi^{r-1}e^{-\xi}d\xi$ is the Euler gamma function. We now want to show that, for large enough s, this formula makes sense and the exchange between the sum and the integral is legitimate. This follows from the next proposition.

Proposition 1.40. *Be^{-tB^2} is a smoothing operator on $L^2(M,S)$ for any $t>0$. Its trace has an asymptotic expansion of the form*

$$\operatorname{tr}(Be^{-tB^2}) \sim \frac{1}{t^{\frac{n}{2}+1}}\sum_{k\geq 0}a_k t^k. \qquad (1.59)$$

Proof. Be^{-tB^2} is clearly smoothing, since the function re^{-tr^2} is rapidly decreasing. Let $\mathcal{F}$ be its smooth kernel, and let $\mathcal{E}$ be the smooth kernel of e^{-tB^2}. We have then

$$\mathcal{F}(t;x,y) = B_x\mathcal{E}(t;x,y).$$

Now, by Proposition 0.16, $\mathcal{E}$ has an asymptotic expansion of the form

$$\mathcal{E}(t;x,y) \sim \frac{1}{t^{\frac{n}{2}}}e^{-\frac{d(x,y)^2}{4t}}\sum_{k\geq 0}\Theta_k(x,y)t^k.$$

Applying B on the x-variable on both sides, we get an asymptotic expansion of $\mathcal{F}$. On the diagonal of $N\times N$, this will be of the form

$$\mathcal{F}(t;x,x) \sim \frac{1}{t^{\frac{n}{2}+1}}\sum_{k\geq 0}\Phi_k(x).$$

The conclusion follows by taking the fibre trace and integrating. □

Corollary 1.41. *The eta function $\eta(s)$ on a closed manifold M^n is analytic for $\operatorname{Re}s > n+1$, where it is given by*

$$\eta(s) = \frac{1}{\Gamma(\frac{s+1}{2})}\int_0^{+\infty} t^{\frac{s+1}{2}-1}\operatorname{tr}(Be^{-tB^2})dt. \qquad (1.60)$$

Proof. In view of Proposition 1.40, the integral on the right in convergent at 0 for $\operatorname{Re} s > n+1$. Convergence at $+\infty$ is easily established for any s. In fact,

$$\int_1^\infty t^{\frac{s+1}{2}-1} \left|\operatorname{tr}(Be^{-tB^2})\right| dt \le \int_1^\infty t^{\frac{s+1}{2}-1} \sum_\lambda |\lambda| e^{-t\lambda^2} dt =$$
$$= \sum_\lambda |\lambda| \int_1^\infty t^{\frac{s+1}{2}-1} e^{-t\lambda^2} dt \le \sum_\lambda |\lambda| \int_1^\infty t^N e^{-t\lambda^2} dt =$$
$$= C \sum_{\lambda\neq 0} \frac{|\lambda|}{\lambda^{2N}} \int_1^\infty e^{-t\lambda^2} dt = C \sum_{\lambda \neq 0} \frac{|\lambda|}{\lambda^{2N+2}} e^{-\lambda} < \infty$$

(here N is any positive integer such that $N \ge (s+1)/2$, and in the next step we integrate by parts N times). Thanks to the same computations, by dominated convergence we can legitimately exchange the series with the integral in (1.58), so (1.60) follows. □

1.3.2 Bismut-Freed property and finiteness of the eta invariant

A priori, (1.60) cannot be used for $\operatorname{Re} s \le n+1$; in particular, it seems unsuitable to prove anything at $s = 0$. In fact, for odd-dimensional manifolds, Bismut and Freed proved a very strong cancellation property for the trace of Be^{-tB^2}, which makes the integral in (1.7) converge absolutely for many more values of s. We will not attempt to prove the result here. As a single remark, we stress the fact that for this property to be true, it is fundamental that the Dirac operator B is the true Dirac operator associated to a Clifford bundle in which the compatibility condition (0.1) is respected (as it is for the signature operator and the spin Dirac operator). There are important natural examples of Dirac operators which are not compatible in the sense of (0.1), such as $\overline{\partial} + \overline{\partial}^*$ on a complex non-K̈hler manifold.

Theorem 1.42. (Bismut-Freed) *Let N^n a closed manifold of odd dimension n, provided with a (compatible) Clifford bundle. Let B the associated Dirac operator on N. Then, all the coefficients a_k for $k < n/2+1$ in the asymptotic expansion*

$$\operatorname{tr}(Be^{-tB^2}) \sim \frac{1}{t^{\frac{n}{2}+1}} \sum_{k\ge 0} a_k t^k$$

vanish. In particular, viewed as a function of t,

$$\operatorname{tr}(Be^{-tB^2}) \in t^{\frac{1}{2}} C^\infty((0,+\infty), \mathbb{R}).$$

Corollary 1.43. *On a closed manifold of odd dimension, formula* (1.60) *gives an analytic continuation of $\eta(s)$ for* $\operatorname{Re} s > -1$. *In particular, the eta invariant $\eta(B) := \eta(0)$ is finite, and it is given by*

$$\eta(B) = \frac{1}{\sqrt{\pi}} \int_0^{+\infty} t^{-\frac{1}{2}} \operatorname{tr}(Be^{-tB^2}) dt. \tag{1.61}$$

Theorem 1.42 appeared originally in [6]. A re-elaborated version of its proof can be found in [16].

Chapter 2

Γ-eta invariants and index theorems for coverings

Suppose that a manifold M is the orbit space of the action of a discrete group Γ on a covering space $\widetilde{M}$ (for example, take $\widetilde{M}$ to be the universal cover of M and Γ the fundamental group $\pi_1(M)$). Useful information about M can be obtained by studying the covering manifold $\widetilde{M}$. However, the manifold $\widetilde{M}$ has the great disadvantage of not being compact whenever Γ is infinite. From the analytical point of view, the following problem arises: if D is a Dirac operator on M, a Dirac operator $\widetilde{D}$ can be defined on $\widetilde{M}$ by lifting all the structure, but, because of noncompactness, $\ker \widetilde{D}$ needs not be finite-dimensional anymore.

In such a situation, index theory may seem to be impracticable. Nonetheless, in [2] Atiyah defined the concept of a "Γ-index" that applies to the above situation, and proved an index theorem now usually referred to as the Atiyah Γ-index theorem (or L^2-index theorem, for reasons that we shall see). The Γ-index is defined by counting the dimensions in a normalised way that takes into account the action of Γ. The right abstract framework for doing so is that of the so-called Hilbert Γ-modules, where a "Γ-dimension" can be defined by the use of a Von Neumann trace.

Atiyah's theorem was just a starting point for the index theory of covering spaces: we will resume the needed properties of Hilbert Γ-modules and then study the L^2-version of all the theory developed in Chapters 0 and 1, including an index theorem for manifolds with boundary and the consequent signature formula.

2.1 Hilbert modules and dimensions

In the first part we report, without proof, a series of functional analytic definitions and elementary results which will be needed in the following. We analise then the concrete situation of Γ-covering spaces. Full proofs and a more detailed treatment can be found in [15] and [24].

2.1.1 Definitions and basic properties

Let Γ a discrete group. A complex Hilbert space V on which Γ acts on the left by isometries is called a *free Hilbert Γ-module* if there is unitary Γ-equivariant isomorphism between V and a tensor product $H \otimes l^2(\Gamma)$, where H is a Hilbert space on which Γ acts trivially. It is called simply a *Hilbert Γ-module* if there is an isometrical Γ-invariant injection into some $l^2(\Gamma) \otimes H$. A *map* or *morphism* between two Hilbert Γ-modules V_1 and V_2 is a bounded linear map which commutes with the Γ-action. We denote by $\mathcal{B}_\Gamma(V)$ the Von Neumann algebra of endomorphisms of a Hilbert Γ-module V, and by $\mathcal{B}_\Gamma(V)^+$ the subset of positive operators.

On a free Hilbert Γ-module $V \cong H \otimes l^2(\Gamma)$, there is a naturally defined application

$$\mathrm{tr}_\Gamma : \mathcal{B}_\Gamma(V)^+ \to [0, +\infty],$$

called Γ-*trace* or *Von Neumann trace*, given by

$$\mathrm{tr}_\Gamma A = \sum_i (A(b_i \otimes e), b_i \otimes e), \tag{2.1}$$

where $\{b_i\}$ is an orthonormal base for H, and $e \in \Gamma \hookrightarrow l^2(\Gamma)$ is the identity element. It is easy to check that the definition does not depend on the choice of the base.

Let V a free Hilbert Γ-module. As for the usual trace, we call $A \in \mathcal{B}_\Gamma(V)$ a Γ-*trace class operator* if $\mathrm{tr}_\Gamma |A| < \infty$ (here, $|A| := \sqrt{A^*A}$). If A is a Γ-trace class operator, even if not positive, the sum in (2.1) is absolutely convergent and defines a Γ-trace

$$\mathrm{tr}_\Gamma A \in \mathbb{C}.$$

Γ-trace class operators form a two-sided ideal in $\mathcal{B}_\Gamma(V)$ and, as for the usual trace, we have formula

$$\mathrm{tr}_\Gamma(AB) = \mathrm{tr}_\Gamma(BA) \tag{2.2}$$

for any A an operator of Γ-trace class and B an endomorphism. Another property of the trace that we will need is the following continuity property, which is proved as the analogous property for the ordinary trace.

Proposition 2.1. *Let A a Γ-trace class operator, and $\{B_k\} \subseteq \mathcal{B}_\Gamma(V)$ a sequence of operators converging weakly to $B \in \mathcal{B}_\Gamma(V)$. Then,*

$$\lim_{k \to +\infty} \mathrm{tr}_\Gamma(AB_k) = \mathrm{tr}_\Gamma(AB).$$

The Γ-trace permits us to define a concept of dimension for Hilbert Γ-modules that generalises the usual complex dimension of vector spaces. If V is any Hilbert Γ-module, we can fix a Γ-injection $j : V \hookrightarrow H \otimes l^2(\Gamma)$ and define the Γ-*dimension* (or *Von Neumann dimension*) of V as

$$\dim_\Gamma V := \mathrm{tr}_\Gamma P_V \in [0, +\infty],$$

where $P_V : H \otimes l^2(\Gamma) \to H \otimes l^2(\Gamma)$ is the orthogonal projection onto the image of j. As one can easily verify, the definition does not depend on the particular injection into a free Hilbert module chosen. When $\Gamma = \{e\}$, Von Neumann dimension coincides with the usual complex dimension. If Γ is finite, we have

$$\dim_\Gamma V = \operatorname{tr}_\Gamma P_V = \frac{1}{|\Gamma|} \operatorname{tr} P_V = \frac{1}{|\Gamma|} \dim V. \tag{2.3}$$

We say that a sequence

$$\to V_{n-1} \xrightarrow{A_{n-1}} V_n \xrightarrow{A_n} V_{n+1} \to$$

of Hilbert Γ-modules and maps between them is *exact* if $\ker A_n = \operatorname{im} A_{n-1}$ for each n and *weakly exact* if $\ker A_n = \overline{\operatorname{im} A_{n-1}}$. Weak exactness is less powerful than true exactness, but still very useful because it allows additivity of Γ-dimension. In fact, the following easy result holds.

Proposition 2.2. *Let $0 \to V \to W \to Z \to 0$ a weakly exact sequence of Hilbert Γ-modules. Then,*

$$\dim_\Gamma W = \dim_\Gamma V + \dim_\Gamma Z.$$

2.1.2 L^2-spaces of Γ-manifolds

We define a *Riemannian Γ-manifold* to be a Riemannian manifold $\widetilde{M}$ on which the discrete group Γ acts (on the right) freely and properly discontinuously by isometries, so that $\widetilde{M}$ is a Riemannian covering for the orbit space M. Whenever M is compact, we say that $\widetilde{M}$ is a *cocompact* Γ-manifold.

For a cocompact Γ-manifold $\widetilde{M}$, it is easy to prove the existence of *fundamental domains*, that is, subsets $F \subseteq \widetilde{M}$ such that

a) $F \cap \gamma F = \emptyset$ for any nontrivial $\gamma \in \Gamma$;

b) $\bigcup_{\gamma \in \Gamma} gF$ differs from $\widetilde{M}$ only by a measure zero set.

In particular, for a fundamental domain F, we have $L^2(F) \cong L^2(M)$.

If $\widetilde{M}$ admits a fundamental domain, we can obviously write:

$$L^2(\widetilde{M}) \cong \bigoplus_{\gamma \in \Gamma} L^2(\gamma F) \cong l^2(\Gamma) \otimes L^2(F) \cong l^2(\Gamma) \otimes L^2(M).$$

This exhibits $L^2(\widetilde{M})$ as a Hilbert Γ-module, where the action of Γ is given by

$$(\gamma f)(x) := f(x\gamma^{-1}).$$

The same conclusions remain true if we replace $L^2(\widetilde{M})$ with the space of L^2-sections of a Hermitian vector bundle $\widetilde{S}$ which is the lift of a vector bundle S on M.

On a noncompact manifold, a linear operator on $L^2(\widetilde{M}, \widetilde{S})$ which is represented by a smooth integral kernel needs not even be bounded. To bypass this complication, we define a *smoothing operator* A by requiring that it has kernel $\mathcal{K}$ satisfying the following conditions:

- $\mathcal{K}$ is smooth on $\widetilde{M} \times \widetilde{M}$;
- the maps $x \mapsto \mathcal{K}(x, \cdot)$ and $y \mapsto \mathcal{K}(\cdot, y)$ are C^∞ as maps of $\widetilde{M}$ to $L^2(\widetilde{M}, \widetilde{S})$;
- there is an absolute constant $C > 0$ such that

$$\int_{\widetilde{M}} |\mathcal{K}(x,y)|^2 dx < C, \quad \int_{\widetilde{M}} |\mathcal{K}(x,y)|^2 dy < C.$$

When these hypothesis are satisfied, integration under the integral sign is allowed and the operator A actually takes values in C^∞ sections. We are of course interested in the case where A is Γ-equivariant (i.e. $A(\gamma f) = \gamma A f$ for $f \in L^2(\widetilde{M}, \widetilde{S})$, $\gamma \in \Gamma$) and hence it is an endomorphism of $L^2(\widetilde{M}, \widetilde{S})$ as a Hilbert Γ-module.

If A is an equivariant smoothing operator, then, it automatically is of Γ-trace class and its Γ-trace can be computed as

$$\operatorname{tr}_\Gamma A = \int_F \operatorname{tr} \mathcal{K}(x,x) dx, \tag{2.4}$$

where F is a fundamental domain for $\widetilde{M}$. This integral does not depend on the choice of F, since equivariance of A clearly implies $\mathcal{K}(x\gamma, x\gamma) = k(x,x)$ for any $x \in \widetilde{M}, \gamma \in \Gamma$.

2.2 Atiyah's Γ-index theorem

Suppose that $\widetilde{M}$ is a cocompact Riemannian Γ-manifold, with orbit space M. If Γ is a finite group of order m and M is without boundary, then $\widetilde{M}$ is a closed manifold as well. Lifting a graded Dirac operator D on M, we obtain a graded Γ-equivariant Dirac operator $\widetilde{D}$ on $\widetilde{M}$. By the Atiyah-Singer index theorem, then, we get:

$$\operatorname{ind} \widetilde{D}^+ = \int_{\widetilde{M}} \widetilde{AS} = m \int_M AS = m \operatorname{ind} D^+. \tag{2.5}$$

If Γ is the deck transformation group of the covering, it would be natural to define, when possible, $\operatorname{ind}_\Gamma \widetilde{D}^+ = \dim_\Gamma \ker \widetilde{D}^+ - \dim_\Gamma \ker \widetilde{D}^-$. In our current example, Γ is finite of order m and the manifold is closed, hence everything makes sense and, by (2.3), we have

$$\operatorname{ind}_\Gamma \widetilde{D}^+ = \frac{1}{m} \operatorname{ind} \widetilde{D}^+. \tag{2.6}$$

Combining (2.5) and (2.6), we get

$$\operatorname{ind}_\Gamma \widetilde{D}^+ = \operatorname{ind} D^+. \tag{2.7}$$

The aim of this section is to prove that the Γ-index ind_Γ is still well defined when Γ is infinite, and that formula (2.7) remains true in this general situation.

2.2.1 Sobolev spaces on the noncompact covering space

Let us allow Γ to be infinite. As a consequence, $\widetilde{M}$ will be noncompact in general. Is S is a Clifford bundle on M, using the covering map $\widetilde{M} \to M$ we can lift it to a Clifford bundle $\widetilde{S}$ on $\widetilde{M}$. Consequently, we will have a Γ-equivariant Dirac operator $\widetilde{M}$, which is the lift of the Dirac operator on M.

For convenience, we define the Sobolev space $W^s(\widetilde{M}, \widetilde{S})$ to be the closure of $C_c^\infty(\widetilde{M}, \widetilde{S})$ in the norm

$$\|f\|_{W^s}^2 := \|f\|_{L^2}^2 + \|\widetilde{D}f\|_{L^2}^2 + \cdots + \|\widetilde{D}^s f\|_{L^2}^2$$

The following Sobolev embedding theorem holds:

Proposition 2.3. *Let $n = \dim \widetilde{M}$, and $p > n/2$. Then, for any $r \in \mathbb{N}$, there is a continuous embedding*

$$W^{r+p}(\widetilde{M}, \widetilde{S}) \hookrightarrow BC^r(\widetilde{M}, \widetilde{S}),$$

where $BC^r(\widetilde{M}, \widetilde{S})$ is the Banach space of bounded C^r sections with bounded derivatives.

Proof. For any $x \in \widetilde{M}$, take a bump function φ_x on $\widetilde{M}$ with support contained in a fundamental domain for the Γ-action, and which is constantly 1 in a neighbourhood of x. We can choose those φ_x such that their first $r + p$ derivatives $\nabla^i \varphi_x$ are uniformly bounded by an absolute constant. As a consequence, thanks to the geometric properties of Dirac operators, for any $f \in C_c^\infty(\widetilde{M}, \widetilde{S})$ we get an estimate

$$\|\varphi_x f\|_{W^{r+p}} \leq C\|f\|_{W^{r+p}}, \tag{2.8}$$

where C is constant depending on the norms of the first $r+p$ derivatives of φ_x, and therefore it can be chosen uniformly on x as well. Now, since the support of φ_x is contained in a fundamental domain, the section $\varphi_x f$ can be identified with a section $\pi_*(\varphi_x f) \in C^\infty(M, S)$ by the covering map π. By the Sobolev embedding on M, we have hence

$$\|\varphi_x f\|_{C^r} = \|\pi_*(\varphi_x f)\|_{C^r} \leq C'\|\pi_*(\varphi_x f)\|_{W^{r+p}} = C'\|\varphi_x f\|_{W^{r+p}},$$

which, together with (2.8), gives

$$\|\varphi_x f\|_{C^r} \leq C''\|\varphi_x f\|_{W^{r+p}}.$$

The result follows since φ_x is constantly 1 near x. ∎

2.2.2 Self-adjointness of the lifted Dirac operator

Denote by $\widetilde{\mathcal{D}}$ the closure in $L^2(\widetilde{M}, \widetilde{S})$ of $\widetilde{D}$, defined at first on C_c^∞. We want to prove that $\widetilde{\mathcal{D}}$ is a self-adjoint operator, generalising the analogous result for closed manifolds. To do so, we will make use of a parametrix for $\widetilde{D}$.

Proposition 2.4. *There exists a* Γ*-equivariant operator*

$$\widetilde{R} : C^\infty(\widetilde{M}, \widetilde{S}) \to C^\infty(\widetilde{M}, \widetilde{S})$$

such that:

(a) the operators $\widetilde{D}\widetilde{R} - 1$ *and* $\widetilde{R}\widetilde{D} - 1$ *are smoothing;*

(b) restriction of $\widetilde{R}$ *to compactly supported sections extends to a bounded operator* $\widetilde{R} : W^s \to W^{s+1}$ *for any* s*.*

Moreover, $\widetilde{R}$ *can be chosen to be formally self-adjoint.*

The idea to construct $\widetilde{R}$ is to lift a parametrix R for D, so that

$$\begin{cases} DR - 1 = S_1 \\ RD - 1 = S_2 \end{cases} \implies \begin{cases} \widetilde{D}\widetilde{R} - 1 = \widetilde{S}_1 \\ \widetilde{R}\widetilde{D} - 1 = \widetilde{S}_2. \end{cases}$$

The problem is that lifting is not always possible, and even when it is, the lifted operators may lose the property of being smoothing. This happens for example for the parametrix R constructed in Chapter 0 (also known as the *Green operator* for D), where $S_1 = S_2 =$projection onto $\ker D$. To prove Proposition 2.4 we need thus the existence of an *almost local* parametrix R in the closed case. This is normally constructed as a *pseudo-differential* operator (see [13]). We shall take the result for granted without proof.

It is now easy to establish self-adjointness.

Proposition 2.5. *The unbounded operator* $\widetilde{\mathcal{D}}$ *is self-adjoint, and its domain coincide with* $W^1(\widetilde{M}, \widetilde{S})$*.*

Proof. By definition of closure and of Sobolev spaces on $\widetilde{M}$, the domain of $\widetilde{\mathcal{D}}$ is $W^1(\widetilde{M}, \widetilde{S})$. By formal-adjointness, it is clear at this point that the adjoint operator $\widetilde{\mathcal{D}}^*$ has a larger domain than $W^1(\widetilde{M}, \widetilde{S})$ and, on this space, it coincides with $\widetilde{\mathcal{D}}$. To conclude, we need to prove that $\operatorname{Dom} \widetilde{\mathcal{D}}^* \subseteq W^1(\widetilde{M}, \widetilde{S})$. Let hence $f \in \operatorname{Dom} \widetilde{\mathcal{D}}^*$. By definition, it exists $g \in L^2(\widetilde{M}, \widetilde{S})$ such that

$$(f, \widetilde{\mathcal{D}}h)_{L^2} = (g, h)_{L^2} \quad \text{for all } h \in W^1(\widetilde{M}, \widetilde{S}) = \operatorname{Dom} \widetilde{\mathcal{D}}.$$

Let now h be any L^2-section. Using a formally self-adjoint parametrix $\widetilde{R}$, we see that $h = \widetilde{\mathcal{D}}\widetilde{R}h - \widetilde{S}_1 h$, with $\widetilde{S}_1$ a smoothing operator. In particular, for any $h \in L^2(\widetilde{M}, \widetilde{S})$, we can write

$$\begin{aligned}(f, h)_{L^2} &= (f, \widetilde{\mathcal{D}}\widetilde{R}h)_{L^2} - (f, \widetilde{S}_1 h)_{L^2} = (g, \widetilde{R}h)_{L^2} - (f, \widetilde{S}_1 h)_{L^2} = \\ &= (\widetilde{R}g, h)_{L^2} - (\widetilde{S}_1 f, h)_{L^2}.\end{aligned}$$

As a consequence, $f = \widetilde{R}g - \widetilde{S}_1 f \in W^1(\widetilde{M}, \widetilde{S})$. □

2.2.3 Finiteness of the Γ-index

Suppose that $S = S^+ \oplus S^-$ is graded. The grading is kept in the lifting of the structure to $\widetilde{M}$, so that

$$D = \begin{pmatrix} 0 & \widetilde{D}^- \\ \widetilde{D}^+ & 0 \end{pmatrix}$$

is a graded Dirac operator on $\widetilde{M}$. We would like to define $\operatorname{ind}_\Gamma \widetilde{D}^+ = \dim_\Gamma L^2 \ker \widetilde{D}^+ - \dim_\Gamma L^2 \ker \widetilde{D}^-$. To do so, we need to show that these L^2-null-spaces are closed Γ-invariant subspaces of $L^2(\widetilde{M}, \widetilde{S})$ with finite Γ-dimension.

To do so, it suffices to prove the same facts for the ungraded operator $\widetilde{D}$. Γ-invariantness of $L^2 \ker \widetilde{D}$ is clear. Closedness will follow once we have proven equality with the null-space of the Hilbert closure $\widetilde{\mathcal{D}}$. Now, since this is a self-adjoint unbounded operator thanks to Proposition 2.5, the functional calculus $\phi(\widetilde{\mathcal{D}})$ can be defined for any measurable bounded function ϕ defined on $\mathbb{R}$. The following very useful result holds.

Proposition 2.6. *If $\phi : \mathbb{R} \to \mathbb{C}$ is rapidly decreasing, in the sense that $|\phi(|t|)| = O(|t|^{-k})$ for each k, then $\phi(\widetilde{\mathcal{D}})$ is a Γ-equivariant smoothing operator.*

We first prove a technical lemma.

Lemma 2.7. *Let A a bounded self-adjoint operator on $L^2(\widetilde{M}, \widetilde{S})$, and suppose that A maps L^2 boundedly to BC^r for each r. Then, A^2 is a smoothing operator.*

Proof. For notational simplicity, we only prove the theorem for the space of complex-valued functions $L^2(\widetilde{M})$. Since $A : L^2 \to BC^0$ is bounded (let us say by $C \geq 0$), for any fixed $x \in \widetilde{M}$ the linear functional $f \to Af(x)$ is continuous. By the Riesz lemma, it is represented by a function $g_x \in L^2(\widetilde{M})$ with $\|g_x\|_{L^2} < C$. This means that

$$Af(x) = \int_{\widetilde{M}} f(y)\overline{g_x(y)}dy$$

or, equivalently, that the kernel $\mathcal{K}_A$ of A is given by $\mathcal{K}_A(x, y) = \overline{g_x(y)}$.

By self-adjointness, we see that $g_x(y) = \overline{g_y(x)}$. The kernel of A^2 is thus given by

$$\mathcal{K}_{A^2}(x, y) = \int_{\widetilde{M}} \mathcal{K}_A(x, z)\mathcal{K}_A(z, y)dz = \int_{\widetilde{M}} \overline{g_x(z)g_z(y)}dz =$$
$$= \int_{\widetilde{M}} g_y(z)\overline{g_x(z)}dz = Ag_y(x).$$

Now, the map $y \to g_y$ can be easily proved to be a C^∞ map to the Banach space $L^2(\widetilde{M})$. As an example, we show continuity. For any f, using the mean value theorem, we have

$$|(f, g_x - g_y)| = |Af(x) - Af(y)| \leq \sup |\nabla f| d(x, y) \leq C' \|f\|_{L^2} d(x, y),$$

so that $\|g_x - g_y\|_{L^2} \leq Cd(x,y)$. Hence, by hypothesis, the map

$$y \mapsto Ag_y = \mathcal{K}_{A^2}(\cdot, y)$$

is a smooth of $\widetilde{M}$ map into $BC^\infty(\widetilde{M})$, from which it follows that the kernel of A^2 is smooth on $\widetilde{M} \times \widetilde{M}$. Moreover, we have a uniform bound

$$|\mathcal{K}_{A^2}(x,y)| = |Ag_y(x)| \leq C\|g_y\|_{L^2} \leq C^2.$$

The last condition to prove for A^2 to be smoothing is the smoothness of the maps $x \to \mathcal{K}_{A^2}(x, \cdot)$ and $y \to \mathcal{K}_{A^2}(\cdot, y)$ into $L^2(\widetilde{M})$. Now, the second map is smooth because $y \to g_y$ is so; as consequence, the first map is smooth as well thanks to self-ajointeness of A^2. □

Proof. (*Proposition 2.6*) Without loss of generality, we can clearly suppose that ϕ is a non-negative function. Hence, ϕ is the square of $\psi = \sqrt{\phi}$, which is still rapidly decreasing. By functional calculus, we can now define a bounded operator $\widetilde{\mathcal{D}}^r \psi(\widetilde{\mathcal{D}})$ for any r. Thanks to the Sobolev embedding proved in Proposition 2.3, we have thus estimates

$$\|\psi(\widetilde{\mathcal{D}})\|_{BC^r} \leq C(\|\psi(\widetilde{\mathcal{D}})\|_{W^{r+p}} \leq CC'\|f\|_{L^2},$$

so that $\psi(\widetilde{\mathcal{D}})$ maps L^2 boundedly to BC^r for any r. By Lemma 2.7, finally, $\phi(\widetilde{\mathcal{D}}) = \psi(\widetilde{\mathcal{D}})^2$ is smoothing. □

Corollary 2.8. *The null space* $\ker \widetilde{\mathcal{D}}$ *is made of smooth sections and it coincides with* $L^2 \ker \widetilde{D}$. *In particular,* $L^2 \ker \widetilde{D}$ *is closed. Moreover,*

$$\dim_\Gamma L^2 \ker \widetilde{D} = \dim_\Gamma \ker \widetilde{\mathcal{D}} < \infty.$$

Proof. The orthogonal projection P onto $\ker \widetilde{\mathcal{D}}$ is given by $P = \chi_{\{0\}}(\widetilde{\mathcal{D}})$, which, by Proposition 2.6 is a Γ-equivariant smoothing operator. In particular:

a) $\ker \widetilde{\mathcal{D}}$ is made of smooth sections;

b) P is a Γ-trace class operator, so that $\dim_\Gamma \ker \widetilde{\mathcal{D}} = \operatorname{tr}_\Gamma P < \infty$.

Now, the inclusion $\ker \widetilde{\mathcal{D}} \subseteq L^2 \ker \widetilde{D}$ is clear thanks to *a*), while the converse follows by self-adjointness, since every L^2-solution of $\widetilde{D}$ is clearly in the domain of $\widetilde{\mathcal{D}}^*$. □

In the graded situation, denoting by $\widetilde{\varepsilon}$ the grading operator on the graded Clifford bundle $\widetilde{S}$, we can the define the Γ-supertrace of a Γ-trace class operator A on $L^2(\widetilde{M}, \widetilde{S})$ as

$$\operatorname{str}_\Gamma A := \operatorname{tr}_\Gamma(\widetilde{\varepsilon} A).$$

Thanks to our last corollary, on a cocompact Γ-manifold without boundary we can define the Γ-*index* of $\widetilde{D}^+$ as

$$\begin{aligned}\operatorname{ind}_\Gamma \widetilde{D}^+ &:= \operatorname{str}_\Gamma P = \dim_\Gamma \ker \widetilde{\mathcal{D}}^+ - \dim_\Gamma \ker \widetilde{\mathcal{D}}^- = \\ &= \dim_\Gamma L^2 \ker \widetilde{D}^+ - \dim_\Gamma L^2 \ker \widetilde{D}^- \in \mathbb{R},\end{aligned}$$

where P is once again the orthogonal projection onto $\ker \widetilde{\mathcal{D}}$.

We remark that abstract notions of "Γ-Fredholmness" and associated index can be given in the context of Hilbert Γ-modules, but we have chosen not to take this point of view. In that context, as in classical Banach space theory, finiteness of the index can be proven abstractly as a consequence of the existence of a parametrix.

2.2.4 Γ-McKean-Singer formula and conclusion

By the functional calculus, the heat operator $e^{-t\widetilde{\mathcal{D}}^2}$ is defined for any $t > 0$ and, by Proposition 2.6 it is a Γ-equivariant smoothing operator. We want to prove that a McKean-Singer-like formula holds in this context. The proof of Proposition 0.15, though, does not generalise to the current situation, since we do not have a discrete spectrum for $\widetilde{D}$. Here, a more abstract method must be used.

Proposition 2.9. *For any $t > 0$, we have*

$$\operatorname{str}_\Gamma e^{-t\widetilde{\mathcal{D}}^2} = \operatorname{ind}_\Gamma \widetilde{D}^+. \tag{2.9}$$

Proof. First of all, we prove that the term on the left does not depend on t. Since the derivative with respect to t of $e^{-t\widetilde{\mathcal{D}}^2}$ is $-\widetilde{\mathcal{D}}^2 e^{-t\widetilde{\mathcal{D}}^2}$, which is still a smoothing operator, we can differentiate the Γ-supertrace, obtaining

$$\frac{d}{dt} \operatorname{str}_\Gamma e^{-t\widetilde{\mathcal{D}}^2} = -\operatorname{str}_\Gamma(\widetilde{\mathcal{D}}^2 e^{-t\widetilde{\mathcal{D}}^2}) = -\operatorname{str}_\Gamma\left((\widetilde{\mathcal{D}} e^{-\frac{t}{2}\widetilde{\mathcal{D}}^2})^2\right). \tag{2.10}$$

Let $A := \widetilde{\mathcal{D}} e^{-\frac{t}{2}\widetilde{\mathcal{D}}^2}$. Since A exchanges $\widetilde{S}^+$ and $\widetilde{S}^-$, we have $\widetilde{\varepsilon}A^2 = A\widetilde{\varepsilon}A$. As a consequence, following the computation in (2.10), we get

$$\frac{d}{dt} \operatorname{str}_\Gamma e^{-t\widetilde{\mathcal{D}}^2} = -\operatorname{tr}_\Gamma(\widetilde{\varepsilon}A^2) = -\frac{1}{2}\operatorname{tr}_\Gamma(\widetilde{\varepsilon}A^2 - A\widetilde{\varepsilon}A) = -\frac{1}{2}\operatorname{tr}_\Gamma[\widetilde{\varepsilon}A, A] = 0$$

thanks to (2.2).

We prove now that $\operatorname{str}_\Gamma e^{-t\widetilde{\mathcal{D}}^2}$ must tend to $\operatorname{str}_\Gamma P = \operatorname{ind}_\Gamma \widetilde{D}^+$ as $t \to \infty$. Since it is a constant function in t, the thesis will follow. From the spectral theorem for unbounded self-adjoint operators, we see that $e^{-t\widetilde{\mathcal{D}}^2}$ converges strongly to $P = \chi_{\{0\}}(\widetilde{\mathcal{D}}^2)$ as $t \to +\infty$, because $e^{-t\xi^2}$ converges pointwise to $\chi_{\{0\}}(\xi)$ on $\mathbb{R}$, with bounded L^∞ norm (see [20]). Thanks to Proposition 2.1, thus (by writing $e^{-t\widetilde{\mathcal{D}}^2} = e^{-\widetilde{\mathcal{D}}^2} e^{-(t-1)\widetilde{\mathcal{D}}^2}$),

$$\lim_{t\to+\infty} \operatorname{str}_\Gamma e^{-t\widetilde{\mathcal{D}}^2} = \operatorname{str}_\Gamma P = \operatorname{ind}_\Gamma \widetilde{D}^+$$

as required. □

We now want to show the equality $\operatorname{ind}_\Gamma \widetilde{D}^+ = \operatorname{ind} D^+ \left(= \int_M AS\right)$ holds. This is a consequence of the following result.

Proposition 2.10. *Let*

$$\mathcal{E}(t;x,y) \sim h(t;x,y) \sum_{k\geq 0} \Theta_k(x,y) t^k$$

be the asymptotic expansion as $t \to 0$ *of the heat kernel* $\mathcal{E}$ *on* M *(see Proposition 0.16). Then, the heat kernel* $\widetilde{\mathcal{E}}$ *on* $\widetilde{M}$ *has an asymptotic expansion*

$$\widetilde{\mathcal{E}}(t;\tilde{x},\tilde{y}) \sim h(t;\tilde{x},\tilde{y}) \sum_{k\geq 0} \widetilde{\Theta}_k(\tilde{x},\tilde{y}) t^k \tag{2.11}$$

as $t \to 0$*. Moreover, if* $\pi : \widetilde{M} \to M$ *is the covering projection, we have*

$$\widetilde{\Theta}_k(\tilde{x},\tilde{x}) = \Theta_k(\pi) \quad \textit{for all } \tilde{x} \in \widetilde{M}.$$

Proof. (*Sketch*) The existence of an asymptotic expansion can be proved in a similar way to the compact case, using Proposition 2.3. The coefficients $\widetilde{\Theta}_k$, calculated on the diagonal, must be the lifts of the corresponding coefficients on M because both of them are algebraic expressions of the metric and connection coefficients. □

We are now ready to precisely state and prove the Γ-index index theorem.

Theorem 2.11. (Atiyah Γ-index index theorem) *Let* $\widetilde{M}$ *a cocompact Riemannian* Γ*-manifold without boundary. Let* D *a graded Dirac operator on the orbit space* M *and let* $\widetilde{D}$ *be its lift to* $\widetilde{M}$*. Then, the* Γ*-index of* $\widetilde{D}^+$ *is well defined, and the equality*

$$\operatorname{ind}_\Gamma \widetilde{D}^+ = \operatorname{ind} D^+$$

holds.

Proof. By Proposition 2.9, computing the Γ-supertrace with the obvious equivalent of (2.4), we get, for any $t > 0$,

$$\operatorname{ind}_\Gamma \widetilde{D}^+ = \operatorname{str}_\Gamma e^{-t\widetilde{\mathcal{D}}^2} = \int_F \operatorname{str}_{\tilde{x}} \widetilde{\mathcal{E}}(t;\tilde{x},\tilde{x}) d\tilde{x},$$

where $F \subseteq \widetilde{M}$ is a fundamental domain for the Γ-action. Thanks to the asymptotic expansion (2.11), with the same simple argument of Corollary 0.19 we see that $\operatorname{ind}_\Gamma \widetilde{D}^+ = 0$ if the dimension n of the manifolds is odd, and that

$$\operatorname{ind}_\Gamma \widetilde{D}^+ = \int_F \operatorname{str}_{\tilde{x}} \widetilde{\Theta}_{n/2}(\tilde{x},\tilde{x}) d\tilde{x}$$

if n is even. In the odd case we have concluded, because the index of D^+ also vanishes; in the even case, we use the Riemannian isometry (modulo measure zero sets) $F \to M$ to get

$$\int_F \operatorname{str}_{\tilde{x}} \widetilde{\Theta}_{n/2}(\tilde{x},\tilde{x}) dx = \int_M \operatorname{str}_x \Theta_{n/2}(x,x) dx = \operatorname{ind} D^+.$$

□

Since Γ-dimensions are real numbers, the fact that the Γ-index is an integer in this situation is remarkable in its own right. The definition of Γ-index, however, may seem a little bit ad hoc for the problem. To stress the fact that Atiyah's theorem has in fact nontrivial consequences outside the realm of Γ-dimensions, we present the following very simple application.

Corollary 2.12. *Under the same hypothesis, suppose that* $\operatorname{ind} D^+ \neq 0$. *Then, the equation* $\widetilde{D} f = 0$ *has nontrivial solutions. If moreover* Γ *is infinite,* $\ker \widetilde{D}$ *is infinite-dimensional.*

Proof. $\operatorname{ind} D^+ \neq 0$ implies $\operatorname{ind}_\Gamma \widetilde{D}^+ \neq 0$, so at least one between $\dim_\Gamma \ker \widetilde{D}^+$ and $\dim_\Gamma \ker \widetilde{D}^-$ is nonzero. The thesis follows by definition of Γ-dimension. □

2.3 The Γ-index theorem for manifolds with boundary

Suppose that $\widetilde{M}$ is a Riemannian cocompact Γ-manifold with nonempty boundary $\widetilde{N}$. $\widetilde{M}$ will be hence the covering space of a manifold with boundary M, and restriction to the boundary gives a Γ-cover $\widetilde{N} \to N := \partial M$. The aim of this section is to prove a Γ-index formula for a lifted Dirac operator $\widetilde{D}$ on $\widetilde{M}$ that generalises at once Atiyah's Γ-index theorem for cocompact Γ-manifolds without boundary and Atiyah-Patodi-Singers index formula for compact manifolds with boundary. We will require a product structure near the boundary, and impose Atiyah-Patodi-Singer boundary conditions for $\widetilde{D}$. The formula is due to Ramachandran ([18]) and reads as follows:

$$\operatorname{ind}_\Gamma \widetilde{D}^+ = \int_M AS - \frac{h_\Gamma(\widetilde{B}) + \eta_\Gamma(\widetilde{B})}{2}. \tag{2.12}$$

Here, $\widetilde{B}$ is the boundary operator and the correction terms are suitable generalisations of those involved in (1.40). The Γ-eta invariant is particularly important for our study because it will appear in the definition of rho invariants in the next chapter.

The methods used in the proof of (2.12) are in great part an adaptation of those used in Chapter 1 for the proof of the Atiyah-Patodi-Singer index theorem. The main problem to circumvent here is that, being the boundary $\widetilde{N}$ noncompact in general, we do not have a basis of eigensections of $\widetilde{B}$ to apply separation of variables. To do so, Ramachandran uses a very general theorem of Browder and Gårding, valid for elliptic self-adjoint operators on open manifolds, which gives a realisation of the spectral theorem through "generalised eigensections". We shall explain how to use the theorem of Browder and Gårding and give a sketch which summarises the proof of the index formula in this context. We conclude the section with a reformulation for manifolds with infinite cylindrical ends. As in Chapter 2, this reformulation will be useful to apply the Γ-index formula to (Γ-)signature problems for manifolds with boundary.

2.3.1 Γ-eta invariants

On a closed odd-dimensional manifold, the eta function of a Dirac operator B was at first defined by

$$\eta(s) = \sum_{\lambda \neq 0} \operatorname{sgn} \lambda \, |\lambda|^{-s}, \tag{1.7}$$

where λ runs among the eigenvalues of B. This definition seems uneasy to generalise to Γ manifolds, since the lifted Dirac operato $\widetilde{B}$ does not have a discrete spectrum. In Section 1.3, however, we gave the following alternative definition for η:

$$\eta(s) = \frac{1}{\Gamma(\frac{s+1}{2})} \int_0^{+\infty} t^{\frac{s+1}{2}-1} \operatorname{tr}(\mathcal{B}e^{-t\mathcal{B}^2})dt. \tag{1.60}$$

It is now natural to try to give a Γ-version of η on a Γ-manifold $\widetilde{N}$ as

$$\eta_\Gamma(s) = \frac{1}{\Gamma(\frac{s+1}{2})} \int_0^{+\infty} t^{\frac{s+1}{2}-1} \operatorname{tr}_\Gamma(\widetilde{\mathcal{B}}e^{-t\widetilde{\mathcal{B}}^2})dt \tag{2.13}$$

(here, $\Gamma(\frac{s+1}{2})$ still denotes the Euler gamma-function and has nothing to do with the discrete group Γ). The next result shows that this is indeed possible. We will see in the following that, as well as its compact counterpart, this function is related to the index problem for Γ-manifolds with boundary.

Proposition 2.13. *The Γ-eta function of* (2.13) *is well defined and analytic for* $\operatorname{Re} s > -1$*. In particular, the Γ-eta invariant*

$$\eta_\Gamma(\widetilde{B}) := \eta_\Gamma(0)$$

is finite, and it is given by

$$\eta_\Gamma(\widetilde{B}) = \frac{1}{\sqrt{\pi}} \int_0^{+\infty} t^{-\frac{1}{2}} \operatorname{tr}_\Gamma(\widetilde{\mathcal{B}}e^{-t\widetilde{\mathcal{B}}^2})dt. \tag{2.14}$$

Proof. Convergence at 0 for $\operatorname{Re} s > 1$ is due to the Bismut-Freed property of $\widetilde{B}$. In fact, similarly as how explained in Proposition 2.10, the asymptotic expansion as $t \to 0$ of the Schwartz kernel of $\widetilde{\mathcal{B}}e^{-t\widetilde{\mathcal{B}}^2}$, calculated on the diagonal, is just a lift of the corresponding one for $\mathcal{B}e^{-t\mathcal{B}^2}$. Thanks to Theorem 1.42, hence, we have

$$\operatorname{tr}_\Gamma(\widetilde{\mathcal{B}}e^{-t\widetilde{\mathcal{B}}^2}) \in t^{\frac{1}{2}} C^\infty((0,+\infty), \mathbb{R}).$$

Convergence at $+\infty$ can be established for any s in a similar way to Corollary 1.41. It suffices to substitute formula

$$\operatorname{tr}(\mathcal{B}e^{-t\mathcal{B}^2}) = \sum_\lambda \lambda e^{-t\lambda^2}$$

by

$$\operatorname{tr}_\Gamma(\widetilde{\mathcal{B}}e^{-t\widetilde{\mathcal{B}}^2}) = \int_{\mathbb{R}} \lambda e^{-t\lambda^2} dm_\Gamma(\lambda),$$

where m_Γ is the measure given by $\Omega \mapsto \operatorname{tr}_\Gamma \chi_\Omega(\widetilde{\mathcal{B}})$. □

2.3.2 Ramachandran Γ-index formula

Let $\widetilde{M}$ is a Riemannian cocompact Γ-manifold with boundary $\widetilde{N} := \partial\widetilde{M}$, provided with a Clifford bundle $\widetilde{S}$ which is the lift of a Clifford bundle S on the orbit space for the Γ-action M. We suppose that all the structure is a product near the boundary. As in the compact case, hence, the positive part of the lifted Dirac operator $\widetilde{D}$ is locally of the form

$$\widetilde{D}^+ = \sigma\left(\frac{\partial}{\partial u} + \widetilde{B}\right),$$

where $\widetilde{B}$ is a Dirac operator on $\widetilde{N}$ (in fact, it is the lift of the boundary operator B for the corresponding formula on M). Thanks to Proposition 2.5, the closure $\widetilde{\mathcal{B}}$ of $\widetilde{B}$ is a self-adjoint operator on $L^2(\widetilde{N}, \widetilde{S}^+|_{\widetilde{N}})$. We define $\widetilde{P}$ to be the spectral projection

$$\widetilde{P} := \chi_{[0,+\infty)}(\widetilde{\mathcal{B}}),$$

and we impose to the index problem the global boundary condition

$$\widetilde{P}f|_{\widetilde{N}} = 0. \tag{2.15}$$

On that regard, we define $C_c^\infty(\widetilde{M}, \widetilde{S}^+; \widetilde{P})$ to be the space of complactly supported positive sections satisfying (2.15). Similarly, we define $C_c^\infty(\widetilde{M}, \widetilde{S}^-; 1-\widetilde{P})$. We are interested in the operators

$$\widetilde{D}^+ : C_c^\infty(\widetilde{M}, \widetilde{S}^+; \widetilde{P}) \to C_c^\infty(\widetilde{M}, \widetilde{S}^-), \quad \widetilde{D}^- : C_c^\infty(\widetilde{M}, \widetilde{S}^-; 1-\widetilde{P}) \to C_c^\infty(\widetilde{M}, \widetilde{S}^+),$$

and to their Hilbert closures in L^2, which we will denote as $\widetilde{\mathcal{D}}^+$ and $\widetilde{\mathcal{D}}^-$ respectively.

The role of eigensection decomposition for B that we used systematically in Chapter 2 will be played by the following result, proved independently by Browder and Gårding, which is valid in the broader context of elliptic operators on manifolds. For a proof, we refer to [12].

Theorem 2.14. (Browder, Gårding) *There exists a possibly infinite sequence of measurable sectional maps $e_j : \mathbb{R} \times N \to \widetilde{S}^+|_{\widetilde{N}}$, smooth on the second variable, and measures μ_j on $\mathbb{R}$ such that the following facts hold.*

1. *For each $\lambda \in \mathbb{R}$ and $x \in \widetilde{N}$, we have*

$$\widetilde{B}e_j(\lambda, x) = \lambda e_j(\lambda, x).$$

2. *There is an isometry*

$$V : L^2(\widetilde{N}, \widetilde{S}) \to \bigoplus_j L^2(\mathbb{R}; \mu_j)$$

explicitly given on a smooth compactly supported section f by

$$(Vf)_j(\lambda) = \int_{\widetilde{N}} (f(x), e_j(\lambda, x))dx.$$

3. *The map V transmorms $\widetilde{B}$ into multiplication by λ, so that*

$$(V(\widetilde{B}f))_j(\lambda) = \lambda(Vf)_j(\lambda)$$

for any $f \in C_c^\infty(\widetilde{N}, \widetilde{S}^+|_{\widetilde{N}})$, and similarly transforms a functional calculus $\phi(\widetilde{\mathcal{B}})$ into multiplication by $\phi(\lambda)$.

4. *If $f \in C_c^\infty(\widetilde{N}, \widetilde{S}^+|_{\widetilde{N}})$, the following representation formula holds:*

$$f(x) = \sum_j \int_{\mathbb{R}} (Vf)_j(\lambda) e_j(\lambda, x) d\mu_j(\lambda).$$

The simplest example to see this theorem in action is for $N = \mathbb{R}$ and $B = \frac{1}{i}\frac{\partial}{\partial x}$. In this case, one single family of generalised eigenfunctions $e^{i\lambda x}$ is sufficient, and the role of V is played by the Fourier transform $(\mathcal{F}f)(\lambda) = \int_{\mathbb{R}} f(x)e^{-i\lambda x}dx$. Of course, $\mathcal{F}$ exchanges B with multiplication by λ and admits the desired inversion formula.

As a first application of Theorem 2.14, we obtain a noncompact trace theorem. Let the Sobolev spaces $W^r(\widetilde{M}, \widetilde{S})$ be defined as in the previous section.

Proposition 2.15. *The restriction to the boundary*

$$C_c^\infty(\widetilde{M}, \widetilde{S}) \to C_c^\infty(\widetilde{N}, \widetilde{S}_{\widetilde{N}})$$
$$f \mapsto f|_{\widetilde{N}}$$

extends to a bounded operator

$$W^1(\widetilde{M}, \widetilde{S}) \to L^2(\widetilde{N}, \widetilde{S}_{\widetilde{N}}).$$

Proof. Thanks to the local product structure, we can subtitute $\widetilde{M}$ with $N \times \mathbb{R}$. We now use Theorem 2.14, by which the $f \in W^1(N \times \mathbb{R}, \widetilde{S})$ are those satisfying

$$\|f\|_{W^1}^2 = \sum_j \int_{\mathbb{R}} \int_{\mathbb{R}} (1 + \lambda^2 + \xi^2) |(\widehat{Vf})_j(\lambda, \xi)|^2 d\mu \, d\lambda < \infty,$$

where $(Vs)_j$ is here a two-variable function because f is a family of functions on $\widetilde{N}$ parametrised by $t \in \mathbb{R}$, and the hat denotes Fourier transform in the t-direction. Classical computations (see [7]) now show that the L^2-norm

$$\left\|f|_{\widetilde{N}}\right\|_{L^2}^2 = \sum_j \int_{\mathbb{R}} |(Vf)_j(\lambda, 0)|^2 d\lambda$$

can be estimated by the W^1-norm. □

We will now state the main theorem of the section. We shall give a detailed sketch of the proof, which is mainly a revision of the whole Section 1.1 in the context of Γ-manifolds with boundary (except for the final reformulation, which for clearness will be studied separately).

Theorem 2.16. *Let $\widetilde{M}$ an even-dimensional cocompact oriented Riemannian Γ-manifold with boundary, and let $\widetilde{S} = \widetilde{S}^+ \oplus \widetilde{S}^-$ a lifted graded Clifford bundle on $\widetilde{M}$, with (Γ-equivariant) graded Dirac operator $\widetilde{D}$. Suppose that, near the boundary $\widetilde{N}$, $\widetilde{M}$ is isometrically diffeomorphic to a product $\widetilde{N} \times [0,1]$, so that the positive part of $\widetilde{D}$ locally takes the special form*

$$\widetilde{D}^+ = \sigma \left(\frac{\partial}{\partial u} + \widetilde{B} \right),$$

where $\sigma : \widetilde{S}^+ \to \widetilde{S}^-$ is Clifford multiplication by the inward normal vector and $\widetilde{B}$ is a Dirac operator on $\widetilde{N}$. We impose on D^+ the boundary condition

$$\widetilde{P} f|_{\widetilde{N}} = 0$$

and, on D^-, the adjoint boundary condition $\sigma^{-1}(1-P)$ (here $\widetilde{P}$ is the spectral projection of $\widetilde{B}$ relative to $[0, +\infty)$). Then, a finite Γ-index

$$\operatorname{ind}_\Gamma \widetilde{D}^+ := \dim_\Gamma L^2 \ker \widetilde{D}^+ - \dim_\Gamma L^2 \ker \widetilde{D}^-$$

can be defined. Moreover, this number can be computed as

$$\operatorname{ind}_\Gamma \widetilde{D}^+ = \int_M AS - \frac{h_\Gamma(\widetilde{B}) + \eta_\Gamma(\widetilde{B})}{2}, \tag{2.16}$$

where

- *$AS := \widehat{A}(TM) \operatorname{ch}_s(S/\Phi)$ is the Atiyah-Singer integrand of Theorem 0.21;*
- *$h_\Gamma(B)$ is the Γ-dimension of the L^2-null space of $\widetilde{B}$;*
- *$\eta_\Gamma(B) = \int_0^{+\infty} t^{-\frac{1}{2}} \operatorname{tr}_\Gamma(\widetilde{\mathcal{B}} e^{-t\widetilde{\mathcal{B}}^2}) dt$ is the Γ-eta invariant of $\widetilde{B}$, which is well defined thanks to 1.12.*

Proof. (*Sketch*) As in the compact case, we want to construct parametrices for the Dirac operator and for the heat operator by patching together a parametrix on the cylinder $\widetilde{C} := \widetilde{N} \times \mathbb{R}^+$ with a parametrix on the boundaryless double $d\widetilde{M}$. The equation $\widetilde{D}^+ f = g$ on $\widetilde{C}$ is solved thanks to Theorem 2.14 by separation of variables: an inverse to $\widetilde{A} := \frac{\partial}{\partial u} + B$ on the cylinder with Atiyah-Patodi-Singer boundary conditions is given by

$$(Qg)(x,u) = \sum_j \int_{\mathbb{R}} h_j(\lambda, u) e_j(\lambda, x) d\mu_j(\lambda),$$

where

$$h_j(\lambda, u) = \begin{cases} \displaystyle\int_0^u e^{\lambda(v-u)} (Vg)_j(\lambda, v) dv, & \text{for } \lambda \geq 0 \\ \displaystyle\int_u^\infty e^{\lambda(v-u)} (Vg)_j(\lambda, v) dv, & \text{for } \lambda < 0, \end{cases}$$

and similarly for an inverse $\widetilde{Q}^*$ of the formal adjoint $\widetilde{A}^* := -\frac{\partial}{\partial u} + B$. Essential adjointness on the cylinder follows, and this permits us to define the cylindrical heat operators $e^{-t\widetilde{\mathcal{A}}^*\widetilde{\mathcal{A}}}$ and $e^{-t\widetilde{\mathcal{A}}\widetilde{\mathcal{A}}^*}$, which are fundamental solutions to the heat equation con $\widetilde{C}$. Again by Theorem 2.14 and separation of variables, we find and explicit expression for the cylindrical heat kernel. In particular, letting

$$\widetilde{\mathcal{K}}(t) := \operatorname{tr}_\Gamma e^{-t\widetilde{\mathcal{A}}^*\widetilde{\mathcal{A}}} - \operatorname{tr}_\Gamma e^{-t\widetilde{\mathcal{A}}\widetilde{\mathcal{A}}^*},$$

with the convention $\operatorname{sign} 0 = 1$ we get

$$\widetilde{\mathcal{K}}(t) = -\frac{1}{2}\int_{\mathbb{R}} \operatorname{sgn}\lambda \operatorname{erfc}(|\lambda|\sqrt{t}) dm_\Gamma(\lambda). \tag{2.17}$$

We construct then parametrices for $\mathcal{D}^+$ and $\widetilde{D}^-$ on $\widetilde{M}$ (with APS boundary conditions) by glueing together the inverses for $\widetilde{A}$ and $\widetilde{A}^*$ with parametrices on the double. As a consequence, we can prove adjointness of the closures $\widetilde{\mathcal{D}}^+$ and $\widetilde{\mathcal{D}}^-$ (see Proposition 1.16). Now, Proposition 2.6 holds in this case, so that we can then define a finite Γ-index by

$$\begin{aligned} \operatorname{ind}_\Gamma \widetilde{D}^+ &:= \dim_\Gamma \ker \widetilde{\mathcal{D}}^+ - \dim_\Gamma \ker \widetilde{\mathcal{D}}^- = \\ &= \dim_\Gamma L^2 \ker \widetilde{D}^+ - \dim_\Gamma L^2 \ker \widetilde{D}^-. \end{aligned}$$

For the same reason, the heat operators $e^{-t\widetilde{\mathcal{D}}^-\widetilde{\mathcal{D}}^+}$ and $e^{-t\widetilde{\mathcal{D}}^+\widetilde{\mathcal{D}}^-}$ are well defined smoothing operators, and McKean-Formula

$$\operatorname{ind}_\Gamma \widetilde{D}^+ = \operatorname{tr}_\Gamma(e^{-t\widetilde{\mathcal{D}}^-\widetilde{\mathcal{D}}^+} - e^{-t\widetilde{\mathcal{D}}^+\widetilde{\mathcal{D}}^-})$$

is proved as in Proposition 2.9. To compute the index, we use an approximate heat kernel constructed by glueing together the cylindrical heat kernels for A and A^* with the heat kernel on the boundaryless double of $\widetilde{M}$, which thanks to Proposition 2.10 is asymptotically just a lift of the heat kernel on the double of $\widetilde{M}$.

Combining everything we did so far, we get, as in Proposition 1.21

$$\operatorname{ind}_\Gamma \widetilde{D}^+ \sim \widetilde{\mathcal{K}}(t) + \sum_{k \geq -2l} t^{\frac{k}{2}} \int_M \alpha_k(z) dz,$$

where $\alpha_k(z)$ are the coefficients in the asymptotic expansion for the fibre supertrace of the heat kernel on $\widetilde{M}$. This last formula gives an asymptotic expansion as $t \to 0$ for $\widetilde{\mathcal{K}}(t)$, in which the constant term is $(\operatorname{ind}_\Gamma \widetilde{D}^+ - \int_M \alpha_0(z)dz)$. From (2.17), with computations very similar to those in Section 1, we find

$$\eta_\Gamma(B) = -\left(h_\Gamma(B) + 2\operatorname{ind}_\Gamma \widetilde{D}^+ - 2\int_M \alpha_0(z)dz\right).$$

Rearranging the terms and identifying the integrand, the result is proven. □

2.3.3 Reformulation for Γ-manifolds with cylindrical ends

From a Riemannian Γ-manifold $\widetilde{M}$, with product structure near the boundary $\widetilde{N}$, we can construct the Riemannian Γ-manifold with cylindrical ends

$$\widetilde{M}_\infty := (\widetilde{N} \times (-\infty, 0]) \cup_{\widetilde{N}} \widetilde{M}.$$

Similarly to what we did in Chapter 1, we want to compare the L^2-null spaces of the graded Dirac operators

$$\widetilde{D}^+_\infty : C^\infty(\widetilde{M}_\infty, \widetilde{S}^+_\infty) \to C^\infty(\widetilde{M}_\infty, \widetilde{S}^-_\infty), \quad \widetilde{D}^-_\infty : C^\infty(\widetilde{M}_\infty, \widetilde{S}^-_\infty) \to C^\infty(\widetilde{M}_\infty, \widetilde{S}^+_\infty)$$

with those of

$$\widetilde{D}^+ : C^\infty(\widetilde{M}, \widetilde{S}^+; \widetilde{P}) \to C^\infty(\widetilde{M}, \widetilde{S}^-), \quad \widetilde{D}^- : C^\infty(\widetilde{M}, \widetilde{S}^-; 1 - \widetilde{P}) \to C^\infty(\widetilde{M}, \widetilde{S}^+)$$

studied above, in order give a reformulation of the Ramachandran index formula in the context of manifolds with infinite cylindrical ends.

Extended L^2-sections on $\widetilde{M}_\infty$ are defined as in Chapter 1: namely, $L^2_{\text{ext}}(\widetilde{M}_\infty, \widetilde{S}_\infty)$ is the space of sections f which are locally in L^2 and such that, for large negative u, they can be written in the form

$$f(x,u) = f_{L^2}(x,u) + f_\infty(y),$$

where $f_{L^2} \in L^2(\widetilde{M}_\infty, \widetilde{S}_\infty)$ and $f_\infty \in L^2(\widetilde{N}, \widetilde{S}|_{\widetilde{N}})$ (this last section is called the *limiting value* of f). The space $L^2_{\text{ext}} \ker \widetilde{D}^\pm_\infty$ is then defined as the space of extended L^2-solutions for $\widetilde{D}^\pm_\infty$. The main result is the following.

Proposition 2.17. *There are natural isomorphisms*

(i) $L^2 \ker \widetilde{D}^+ \cong L^2 \ker \widetilde{D}^+_\infty$,

(ii) $L^2 \ker \widetilde{D}^- \cong L^2_{\text{ext}} \ker \widetilde{D}^-_\infty$.

In particular, the Γ*-dimension of the* L^2*-null spaces of* $\widetilde{D}^+_\infty$ *and* $\widetilde{D}^-_\infty$ *are finite.*

Proof. The proof is the same of that of Proposition 1.23, except that we have to use Theorem 2.14 instead of the eigensection expansion. As an example, we prove (i).

Let $f \in L^2 \ker \widetilde{D}^+$. Thanks to Theorem 2.14, near the boundary we can expand f in the form

$$f(x,u) = \sum_j \int_{\mathbb{R}} (Vf)_j(\lambda, u) e_j(\lambda, x) d\mu_j(\lambda).$$

Since $\widetilde{D}^+ = \sigma(\partial/\partial u + \widetilde{B})$, the equation $\widetilde{D}^+ f = 0$ translates into the family of ordinary differential equations

$$(\partial/\partial u + \lambda)(Vf)_j(\lambda, u) = 0,$$

which have

$$(Vf)_j(\lambda, u) = e^{-\lambda u}(Vf)_j(\lambda, 0)$$

as solutions. The boundary condition $\widetilde{P}f|_{\widetilde{N}} = 0$ implies however that $(Vf)_j(\lambda, 0) = 0$ for any $\lambda \geq 0$. Hence, f extends uniquely to a solution of $\widetilde{D}^+_\infty f = 0$ on the whole $\widetilde{M}_\infty$, which is of the form

$$f(x, u) = \sum_j \int_{(-\infty,0)} e^{-\lambda u}(Vf)_j(\lambda, 0)e_j(\lambda, x)d\mu_j(\lambda)$$

on the infinite cylinder. This clearly belongs to $L^2(\widetilde{M}_\infty, \widetilde{S}^+_\infty)$.

Conversely, any $f \in L^2 \ker \widetilde{D}^+_\infty$, on the cylindrical end will be of the form

$$f(x, u) = f(x, u) = \sum_j \int_{\mathbb{R}} e^{-\lambda u}(Vf)_j(\lambda, 0)e_j(\lambda, x)d\mu_j(\lambda).$$

But, then, all $(Vf)_j(\lambda, 0)$ must hence be 0 for $\lambda \geq 0$, otherwise f would not be in L^2. As a consequence, restriction of f to $\widetilde{M}$ respects the boundary condition and is hence in $L^2 \ker \widetilde{D}^+$.

The Γ-dimension of the L^2-null spaces of $\widetilde{D}^+_\infty$ and $\widetilde{D}^-_\infty$ are finite thanks to 2.16. □

We can now proceed in the exact same manner of Chapter 1 to get the following reformulation of Ramachandran's index formula.

Corollary 2.18. *Let h^+_Γ denote the Γ-dimension of L^2-solutions of $\widetilde{B}$ which are limiting values to extended L^2-solutions of $\widetilde{D}^+_\infty$, and define h^-_Γ accordingly. Then, we have*

$$h^+_\Gamma + h^-_\Gamma = h_\Gamma(\widetilde{B}), \tag{2.18}$$

where, once again, $h_\Gamma(\widetilde{B}) = \dim_\Gamma L^2 \ker \widetilde{B}$. Moreover, the following formula holds:

$$\dim_\Gamma L^2 \ker \widetilde{D}^+_\infty - \dim_\Gamma L^2 \ker \widetilde{D}^-_\infty = \int_M AS + \frac{1}{2}(h^+_\Gamma - h^-_\Gamma - \eta_\Gamma(\widetilde{B})). \tag{2.19}$$

We conclude with one more nontrivial result that will be needed in next section.

Proposition 2.19. *The extended L^2-solutions of D^2_∞ and D_∞ coincide on M_∞. In particular, the true L^2-solutions also coincide for these operators.*

Proof. Once again, this is just a work of translating the proof given in the classical case (Proposition 1.27) into the new contest, using Browder-Gårding's theorem instead of the eigensection expansion. □

2.4 Γ-signature formulas

In case of a noncompact manifold, with the help of a Riemannian structure, a so-called L^2-cohomology can be defined. We shall restrict our attention to the *reduced* version, in which one quotients by a closed subspace and hence cohomology groups are Hilbert spaces. If we apply this procedure to a cocompact Γ-manifold $\widetilde{M}$, the L^2-cohomology will be a Hilbert Γ-module, and numerical invariants such as the L^2-Betti numbers and an L^2-signature can be extracted. After having defined and studied this de Rham L^2-cohomology, we will concentrate to the signature problem, trying to apply our Γ-index theory to obtain signature formulas for Γ-manifolds. We will see that, for manifolds with boundary, this is a little trickier than in the compact case. The hardest step was proved by Lück and Schick in [14].

2.4.1 L^2-de Rham cohomology and Γ-signature

Let X an oriented Riemannian manifold (possibly with boundary), and consider the usual complex of compactly supported differential forms (with complex values):

$$\to \Omega_c^{p-1}(X) \xrightarrow{d_{p-1}} \Omega_c^p(X) \xrightarrow{d^p} \Omega_c^{p+1}(X) \to .$$

Denote by $\mathbf{d}^p$ the Hilbert closure of d^p as an unbounded operator $L^2\Omega^j(X) \to L^2\Omega^{j+1}(X)$ (here the Riemmannian metric enters the game). We have then a sequence of unbounded operators

$$\to L^2\Omega_c^{p-1}(X) \xrightarrow{\mathbf{d}^{p-1}} L^2\Omega_c^p(X) \xrightarrow{\mathbf{d}^p} L^2\Omega_c^{p+1}(X) \to \qquad (2.20)$$

with the property that $\overline{\operatorname{im}\mathbf{d}^p} \subseteq \ker \mathbf{d}^{p+1}$ for each p (the fact that $\operatorname{im}\mathbf{d}^p \subseteq \ker \mathbf{d}^{p+1}$ follows from the definition of Hilbert closure, and $\ker \mathbf{d}^{p+1}$ is a closed subspace of $L^2\Omega^{p+1}(X)$). We define, for any p, the pth group of (reduced)*L^2-de Rham cohomology* of X as

$$H^p_{(2)}(X) := \ker \mathbf{d}^p / \overline{\operatorname{im}\mathbf{d}^{p-1}}.$$

We highlight the fact that this is a *reduced* cohomology theory: we are not quotienting by the image of the differential, but by its closure. A great advantage is that, in doing so, the structure of Hilbert space is mantained by $H^p_{(2)}(X)$.

We recall that, if X is complete and without boundary, it admits a Hodge decomposition

$$L^2\Omega^p(X) = \mathcal{H}^p_{(2)}(X) \oplus \overline{d(\Omega_c^{p-1}(X))} \oplus \overline{\delta(\Omega_c^{p+1}(X))}, \qquad (1.49)$$

where $\mathcal{H}^p_{(2)}(X)$ is the space of L^2-harmonic forms on X. As a consequence, we have the following result.

Proposition 2.20. *If X is an oriented complete Riemannian manifold without boundary. Then, the map*

$$\begin{aligned}\mathcal{H}^*_{(2)}(X) &\to H^*_{(2)}(X)\\ \omega &\mapsto [\omega]\end{aligned} \tag{2.21}$$

is a linear isometry of graded Hilbert spaces.

L^2-cohomology is particularly interesting when the manifold $X = \widetilde{M}$ is a cocompact Riemannian Γ-manifold, covering the compact manifold M. As we saw in 2.1.2, L^2-spaces of differential forms $L^2\Omega^p(\widetilde{M})$ are not only Hilbert spaces, but Hilbert Γ-modules. The de Rham differential d^p, moreover, is a Γ-equivariant map, so that the null space of its closure is a Γ-submodule of $L^2\Omega^p(\widetilde{M})$. The L^2-cohomology group $H^p_{(2)}(\widetilde{M})$ inherits thus the structure of a Hilbert Γ-module. In the context of cocompact Γ-manifolds without boundary, the isometry (2.21) becomes an isometrical isomorphism of Γ-modules. As a consequence, we get the following finiteness property.

Proposition 2.21. *Let $\widetilde{M}$ an oriented cocompact Riemannian Γ-manifold. Then,*

$$\dim_\Gamma H^*_{(2)}(\widetilde{M}) < \infty.$$

Proof. Thanks to Corollary 2.20, we have

$$H^*_{(2)}(\widetilde{M}) = \mathcal{H}_{(2)}(\widetilde{M}) = L^2\ker(d+\delta),$$

and the null space of $d+\delta$ has finite Γ-dimension because $d+\delta$ is a Dirac operator (see Proposition 2.8). □

Useful numerical invariants we can extract from the L^2-cohomology of a cocompact Γ-manifold without boundary are the L^2*-Betti numbers*, which are simply defined as

$$b^p_\Gamma(\widetilde{M}) := \dim_\Gamma H^p_{(2)}(\widetilde{M}) \in [0,+\infty).$$

Taking $\widetilde{M}$ to be the universal cover of M, these numbers are well defined invariants of the underlying manifold M, and can also be denoted by $b^p_{(2)}(M)$. As in the classical case, one can then define the *Euler* Γ*-characteristic* as

$$\chi_\Gamma(\widetilde{M}) := \sum_i (-1)^i b^i_\Gamma(\widetilde{M}).$$

While the L^2-Betti numbers need not be integers and can differ considerably from the ordinary Betti numbers of M, an elementary consequence of the Atiyah Γ-index theorem is that

$$\chi_\Gamma(\widetilde{M}) = \chi(M) \in \mathbb{Z},$$

where $\chi(M)$ denotes the ordinary Euler characteristic of the closed manifold M (to do so, it suffices to identify the two terms as the Γ-index and ordinary index of $d+\delta$ on $\widetilde{M}$ and M respectively, once that we have chosen the grading on forms to be given by the parity).

Suppose now that the dimension of $\widetilde{M}$ is $4k$. We can define a Γ-signature of $\widetilde{M}$ applying the following abstract contruction.

Let V a Hilbert Γ-module, and let

$$b : V \times V \to \mathbb{C}$$

be a continuous Hermitian Γ-invariant form. Then, we can associate to b a self-adjoint map of Hilbert modules $B : V \to V$ uniquely determined by $(Bv, w) = b(v, w)$. V decomposes thus as the direct sum of closed Γ-submodules

$$V = V_0 \oplus V_+ \oplus V_-, \tag{2.22}$$

where $V_0 = \ker B$, $V_+ = \chi_{(0,+\infty)}(B)$ and $V_- = \chi_{(-\infty,0)}(B)$. If V has finite Γ-dimension, the Γ-*signature* of b is then defined as

$$\operatorname{sign}_\Gamma(b) := \dim_\Gamma V_+ - \dim_\Gamma V_-.$$

For $\widetilde{M}$ an oriented cocompact Riemannian Γ-manifold without boundary, we can consider the intersection form given by

$$\begin{aligned} L^2\Omega^{2k}(\widetilde{M}) \times L^2\Omega^{2k}(\widetilde{M}) &\to \mathbb{C} \\ (\alpha, \beta) &\mapsto \int_{\widetilde{M}} \alpha \wedge \overline{\beta} := (\star\alpha, \beta)_{L^2}. \end{aligned}$$

This is obviously Hermitian and Γ-invariant. Moreover, by continuity of the scalar product, it descends to L^2-cohomology, providing a Hermitian form

$$h : H^{2k}_{(2)}(\widetilde{M}) \times H^{2k}_{(2)}(\widetilde{M}) \to \mathbb{C}.$$

We define the Γ-*signature* of $\widetilde{M}$ as the Γ-signature of h, i.e.

$$\operatorname{sign}_\Gamma \widetilde{M} := \operatorname{sign}_\Gamma(h).$$

2.4.2 Atiyah's Γ-signature formula

Let $\widetilde{M}^{4k}$ an oriented cocompact Riemannian Γ-manifold without boundary. We want to compare the Γ-signature of $\widetilde{M}$ with the classical signature of the closed manifold $M = \widetilde{M}/\Gamma$. Thanks to Corollary 2.20, we could as well define the signature of $\widetilde{M}$ through the corresponding Hermitian form

$$h : \mathcal{H}^{2k}_{(2)}(\widetilde{M}) \times \mathcal{H}^{2k}_{(2)}(\widetilde{M}) \to \mathbb{C}.$$

As usual, this permits us to relate the signature with the index of the signature operator $\widetilde{D} = d + \delta$ on $\widetilde{M}$.

Proposition 2.22. *Let $\widetilde{D} = d + \delta$ the signature operator on $\widetilde{M}$. Then*

$$\operatorname{sign}_\Gamma(\widetilde{M}) = \operatorname{ind}_\Gamma \widetilde{D}^+$$

.

Proof. The proof is the same of Proposition 0.13, but we repeat the last part of it in this new setting. Exploiting the right cancellations, we find that

$$\operatorname{ind} \widetilde{D}^+ = \dim_\Gamma(\mathcal{H}^{2k}_{(2)}(\widetilde{M}) \cap \Omega^+) - \dim_\Gamma(\mathcal{H}^{2k}_{(2)}(\widetilde{M}) \cap \Omega^-),$$

here $\Omega^\pm$ denotes the positive/negative L^2-harmonic forms with respect to the signature grading. Now, by definition, the Hodge star operator is equal to ± 1 on $\Omega^\pm$. Let h the intersection pairing on $\mathcal{H}^{2k}_{(2)}(\widetilde{M}) \times \mathcal{H}^{2k}_{(2)}(\widetilde{M})$, for which, thanks to Corollary 2.20, we have $\operatorname{sign}_\Gamma(h) = \operatorname{sign}_\Gamma(\widetilde{M})$. Then,

$$h(\alpha, \beta) = \int_{\widetilde{M}} \star\alpha \wedge \overline{\beta} = (\alpha, \beta)_{L^2} \qquad \text{for } \alpha \in \mathcal{H}^{2k}_{(2)}(\widetilde{M}) \cap \Omega^+$$

and

$$h(\alpha, \beta) = \int_{\widetilde{M}} - \star \alpha \wedge \overline{\beta} = -(\alpha, \beta)_{L^2} \qquad \text{for } \alpha \in \mathcal{H}^{2k}_{(2)}(\widetilde{M}) \cap \Omega^-.$$

This means that the bounded self-adjoint operator associated to h acts as $+1$ on the first subspace and as -1 on the second subspace. It follows that, in the decomposition (2.22), we have $V_0 = 0$ and $V^\pm = \mathcal{H}^{2k}(M) \cap \Omega^\pm$. By definition, thus, the Γ-signature of $\widetilde{M}$ is given by

$$\operatorname{sign}_\Gamma M = \operatorname{sign}_\Gamma(h) = \dim_\Gamma(\mathcal{H}^{2k}_{(2)}(\widetilde{M}) \cap \Omega^+) - \dim_\Gamma(\mathcal{H}^{2k}_{(2)}(\widetilde{M}) \cap \Omega^-) = \operatorname{ind}_\Gamma \widetilde{D}^+.$$

□

Corollary 2.23. (Atiyah's Γ-signature formula) *Let $\widetilde{M}^{4k}$ an oriented cocompact Riemannian Γ-manifold without boundary of dimension $4k$, and let M^{4k} be the orbit space of the Γ-action. Then*

$$\operatorname{sign}_\Gamma(\widetilde{M}) = \operatorname{sign} M. \tag{2.23}$$

Proof. Thanks to Propositions 0.13 and 2.22, we have

$$\operatorname{sign} M = \operatorname{ind} D^+, \qquad \operatorname{sign}_\Gamma \widetilde{M} = \operatorname{ind}_\Gamma \widetilde{D}^+.$$

By Atiyah's Γ-index theorem, the assertion follows. □

This formula comes with a series of nontrivial consequences, since it permits to extend to the Γ-signature all properties we know true for the classical signature. Beyond the integrity of $\operatorname{sign}_\Gamma \widetilde{M}$, which could be a priori any real number, we remark that (2.23) gives a direct proof of independence of this cohomological invariant from the metric and even from the differential structure of the Γ-manifold.

2.4.3 L^2-cohomology on manifolds with boundary

Suppose now that X is an oriented Riemannian manifold with boundary. Then, the L^2-de Rham cohomology $H^*_{(2)}(X)$ is defined as above, but we can also define the *relative L^2-de Rham cohomology* of the couple $(X, \partial X)$ in a very similar manner. Namely, we start from the complex

$$\to \Omega_c^{p-1}(X, \partial X) \xrightarrow{d^{p-1}} \Omega_c^p(X, \partial X) \xrightarrow{d^p} \Omega_c^{p+1}(X, \partial X) \to,$$

then we take closures $\mathbf{d}^p$ in L^2 and we set

$$H^p_{(2)}(X, \partial X) := \ker \mathbf{d} / \overline{\operatorname{im} \mathbf{d}}$$

(here, as usual, $\Omega_c^p(X, \partial X)$ denotes the space of compactly supported differential forms on X whose pullback on ∂X is 0).

We recall that, if X is complete (as a metric space), Theorem 1.29 gives the two following Hodge decompositions.

$$\begin{aligned} L^2\Omega^p(X) &= \mathcal{H}^p_{(2)}(X, \partial X)_{\text{abs}} \oplus \overline{d(\Omega_c^{p-1}(X)} \oplus \overline{\delta(\{\omega \in \Omega_c^{p+1}(X) \mid \star\omega|_{\partial X} = 0\}}, \\ L^2\Omega^p(X) &= \mathcal{H}^p_{(2)}(X, \partial X)_{\text{rel}} \oplus \overline{d(\Omega_c^{p-1}(X, \partial X))} \oplus \overline{\delta(\Omega_c^{p+1}(X))}, \end{aligned}$$

where abs and rel denote the absolute and relative boundary conditions on ∂X (see (1.50)).

As a consequence, we get the following isomorphisms for absolute and relative L^2-cohomology.

Proposition 2.24. *Let X be a complete oriented Riemannian manifold with boundary Y. Then, the inclusion of harmonic forms induces isometries*

$$\begin{aligned} \mathcal{H}^*_{(2)}(X, Y)_{\text{abs}} &\cong H^*_{(2)}(X) \\ \mathcal{H}^*_{(2)}(X, Y)_{\text{rel}} &\cong H^*_{(2)}(X, Y). \end{aligned} \tag{2.24}$$

Notice that the $\star$ operator exchanges absolute and relative boundary conditions. Hence, we easily get to the following form of Poincaré duality, which will be useful in the signature problem.

Proposition 2.25. *The $\star$ operator $\star : \mathcal{H}^p_{(2)}(X, Y)_{\text{abs}} \xrightarrow{\sim} \mathcal{H}^{n-p}_{(2)}(X, Y)_{\text{rel}}$ (where $n := \dim X$) can be tranferred to an isometry*

$$H^p_{(2)}(X) \xrightarrow{\sim} H^{n-p}_{(2)}(X, Y).$$

We now pass to analyse the case of an oriented cocompact Riemannian Γ-manifold $\widetilde{M}$ with boundary $\widetilde{N}$. L^2-cohomology groups and spaces of L^2-harmonic forms are then Hilbert Γ-modules, and all isomorphisms seen so far become Γ-equivariant. A fundamental tool is the *long weakly exact L^2-cohomology sequence* of the couple $(\widetilde{M}, \widetilde{N})$, which is constructed by a minor modification of the classical procedure (for details, see [1]).

Proposition 2.26. *There exists a natural weakly exact sequence of Hilbert* Γ*-modules*

$$\to H^{p-1}_{(2)}(\widetilde{N}) \xrightarrow{\partial^*} H^p_{(2)}(\widetilde{M}, \widetilde{N}) \xrightarrow{p^*} H^p_{(2)}(\widetilde{M}) \xrightarrow{j^*} H^p_{(2)}(\widetilde{N}) \xrightarrow{\partial^*} H^{p+1}_{(2)}(\widetilde{N}) \to \tag{2.25}$$

We would now like to define the concept of Γ-signature for a $4k$-dimensional oriented cocompact Riemannian Γ-manifold with boundary. If $\widetilde{M}^{4k}$ is such a manifold (with boundary $\mathcal{N}$), the intersection form

$$s : H^{2k}_{(2)}(\widetilde{M}, \widetilde{N}) \times H^{2k}_{(2)}(\widetilde{M}, \widetilde{N}) \to \mathbb{C}$$

is well defined thanks to integration by parts. The Hilbert Γ-module $H^{2k}_{(2)}(\widetilde{M}, \widetilde{N})$ is then split as

$$H^{2k}_{(2)}(\widetilde{M}, \widetilde{N}) = V_0 \oplus V_+ \oplus V_-,$$

as in (2.22). We would like to define the Γ-signature $\operatorname{sign}_\Gamma(\widetilde{M})$ as

$$\operatorname{sign}_\Gamma(\widetilde{M}) := \dim_\Gamma V_+ - \dim_\Gamma V_-, \tag{2.26}$$

but with the tools at hand we cannot prove any finiteness condition for the Γ-dimension of $H^{2k}_{(2)}(\widetilde{M}, \widetilde{N})$, so the terms in the difference may a priori be infinite (Ramachandran's results cannot be applied because here we hare dealing with *local* boundary conditions). We will show in the next paragraph that at least V_+ and V_- are finite-Γ-dimensional, so that the definition (2.26) makes sense.

2.4.4 The Γ-signature formula for manifolds with boundary

We would like to apply Ramachandran's Γ-index theorem for manifolds with boundary to the signature operator, in order to obtain finiteness results and a Γ-signature formula.

First of all, we find useful to recall the strategy used by Atiyah, Patodi and Singer for a compact manifold M with boundary N, which is the content of our section 1.2.

1. The signature of M is originally defined as the signature of the Hermitian form
$$s : H^{2k}(M, N) \times H^{2k}(M, N) \to \mathbb{C}.$$
Denoting by $\widehat{H}^*(M)$ the image of $\varphi : H^*(M, N) \to H^*(M)$, we see that s descends to
$$\widehat{s} : \widehat{H}^{2k}(M) \times \widehat{H}^{2k}(M) \to \mathbb{C},$$
so that
$$\operatorname{sign} M = \operatorname{sign} \widehat{s}.$$

2. By a cylindrical Hodge theorem, there is a natural isomorphism $\mathcal{H}^*_{(2)}(M_\infty) \cong \widehat{H}^{2k}(M)$, which transforms $\widehat{s}$ into the intersection form

$$s_\infty : \mathcal{H}^*_{(2)}(M_\infty) \times \mathcal{H}^*_{(2)}(M_\infty) \to \mathbb{C}$$

on harmonic L^2-forms on the elongated manifold M_∞. Hence,

$$\operatorname{sign} \widehat{s} = \operatorname{sign} s_\infty.$$

3. By standard techniques, first seen in the proof of Proposition 0.13, we find an equality

$$\operatorname{sign} s_\infty = \dim L^2 \ker D^+_\infty - \dim L^2 \ker D^-_\infty.$$

4. By the reformulation of the Atiyah-Patodi-Singer index formula for manifolds with infinite cylindrical ends, we find

$$\dim L^2 \ker D^+_\infty - \dim L^2 \ker D^-_\infty = \int_M L(M) + \frac{1}{2}(h^+ - h^- - \eta(B)).$$

5. Using exact sequences and duality theorems in homology and cohomology (and the Hodge theorem on N), we find the equality $h^+ = h^-$ so that, finally, we have

$$\operatorname{sign} M = \int_M L(M) - \frac{1}{2}\eta(B).$$

Going back to the context of Γ-manifolds, our goal is to prove the following theorem.

Theorem 2.27. *Let $\widetilde{M}$ be an oriented cocompact Riemannian Γ-manifold with boundary $\widetilde{N}$ and product structure near the boundary. Then, the Γ-signature of $\widetilde{M}$ is well defined and it is given by*

$$\operatorname{sign}_\Gamma \widetilde{M} = \int_M L(M) - \frac{1}{2}\eta_\Gamma(\widetilde{B}),$$

where B is the odd signature operator on $\widetilde{N}$.

To prove Theorem 2.27, we will try to imitate, in the L^2-setting, the above steps 1-5. Some technical problems arise, but the global structure of the proof is mantained. We will use the notation $\operatorname{sign}_\Gamma$ and write equalities between Γ-signatures even when we still do not know whether the necessary finiteness conditions are verified. What we mean is that there are weak isomorphisms between the corresponding V_+'s and V_-'s in the Hilbert Γ-module space decompositions. As such, if one of the two Γ-signatures is well defined, the other is well defined as well, and their values coincide.

Proof. 1. $\operatorname{sign}_\Gamma \widetilde{M}$ is defined as the abstract Γ-signature of

$$s : H^{2k}_{(2)}(\widetilde{M}, \widetilde{N}) \times H^{2k}_{(2)}(\widetilde{M}, \widetilde{N}) \to \mathbb{C}$$
$$([\alpha]_\partial, [\beta]_\partial) \mapsto (\star\alpha, \beta)_{L^2}.$$

As in the compact case, s algebraically descends to the image of $[i]^{2k}$: $H^{2k}_{(2)}(\widetilde{M}, \widetilde{N}) \to H^{2k}_{(2)}(\widetilde{M})$. However, this space needs not be closed, so we cannot compute Γ-signatures on it. We define then

$$\widehat{H}^*_{(2)}(\widetilde{M}) := \overline{\operatorname{im}[i]}.$$

By continuity of s, the bilinear form on $\operatorname{im}[i]^{2k}$ can be extended to

$$\widehat{s} : \widehat{H}^*_{(2)}(\widetilde{M}) \times \widehat{H}^*_{(2)}(\widetilde{M}) \to \mathbb{C}.$$

Thank to Proposition 2.2, hence, we have

$$\operatorname{sign}_\Gamma \widetilde{M} = \operatorname{sign}_\Gamma s = \operatorname{sign}_\Gamma \widehat{s}.$$

2. Consider the space of L^2-harmonic forms on the elongated manifold $\widetilde{M}_\infty$. We get a Hermitian form

$$s_\infty : \mathcal{H}^{2k}_{(2)}(\widetilde{M}_\infty) \times \mathcal{H}^{2k}_{(2)}(\widetilde{M}_\infty) \to \mathbb{C}$$

defined in the usual way. In fact, there is a weak isomorphism between $\mathcal{H}^{2k}_{(2)}(\widetilde{M}_\infty)$ and $\widehat{H}^*_{(2)}(\widetilde{M})$, for which

$$\operatorname{sign}_\Gamma \widehat{s} = \operatorname{sign}_\Gamma s_\infty.$$

This is the most technical step and we will prove it separately.

3. Consider the signature operator $\widetilde{D}_\infty$ on $\widetilde{M}_\infty$. With the same techinique of Proposition 0.13, and the same (mainly notational) L^2-modifications adopted in Proposition 2.22, we get

$$\operatorname{sign}_\Gamma s_\infty = \dim_\Gamma L^2 \ker \widetilde{D}^+_\infty - \dim_\Gamma L^2 \ker \widetilde{D}^-_\infty.$$

The terms in the difference represent the decomposition $V = V_0 \oplus V_+ \oplus V_-$ (with $V_0 = 0$) and, as such, we finally see that the signature is finite thanks to Proposition 2.17 and the subsequent remark.

4. Here we directly apply the reformulation of Ramachandran's index formula given in Corollary 2.18 for Γ-manifolds with infinite cylindrical ends. Namely,

$$\dim L^2 \ker \widetilde{D}^+_\infty - \dim L^2 \ker \widetilde{D}^-_\infty = \int_M L(M) + \frac{1}{2}(h^+_\Gamma - h^-_\Gamma - \eta_\Gamma(\widetilde{B})),$$

where $\widetilde{B}$ is the odd signature operator on $\widetilde{N}$.

5. In order to conclude, we need only prove the equality $h_\Gamma^+ = h_\Gamma^-$. To do so, we will rewrite the proof of Proposition 1.36 in the context of L^2-cohomology.

   Let $\mathcal{K}^*(\widetilde{M}_\infty) := L^2_{\text{ext}} \ker \widetilde{D}_\infty = L^2_{\text{ext}} \ker \widetilde{\Delta}_\infty$ be the space of extended L^2-harmonic forms (the second equality is true thanks to Proposition 2.19), and $\mathcal{K}^\pm(\widetilde{M}_\infty)$ be its graded parts. An element ω of $\mathcal{K}^\pm(\widetilde{M}_\infty)$ is of the form

$$\omega = \omega_{L^2} + \omega_\infty \pm \tau\omega_\infty$$

   (where τ is the grading operator), with $\omega_{L^2} \in \mathcal{H}^*_{(2)}(\widetilde{M}_\infty)$ and $\omega_\infty \in L^2 \ker \widetilde{B} \subseteq \Omega^*(\widetilde{N})$. The number $h_\Gamma^\pm$, by definition, is equal to the Γ-dimension of the image of

$$\begin{aligned} \Pi^\pm : \mathcal{K}^\pm(\widetilde{M}_\infty) &\to L^2 \ker \widetilde{B} \\ \omega &\mapsto \omega_\infty. \end{aligned}$$

   This map can be factorised as

$$\mathcal{K}^\pm(\widetilde{M}_\infty) \to H^*_{(2)}(\widetilde{M}) \xrightarrow{j^*} H^*_{(2)}(\widetilde{N}) \xrightarrow{\sim} \ker L^2 \widetilde{B},$$

   where the first map is $\omega \mapsto [\omega|_{\widetilde{M}}]$, the second is the natural map and the third is the inverse of the Hodge-Kodaira isomorphism (2.21) (after having identified $L^2 \ker \widetilde{B}$ with $\mathcal{H}^*_{(2)}(\widetilde{N})$ as in Remark 1.31). In particular,

$$h_\Gamma^\pm = \dim_\Gamma \operatorname{im} \Pi^\pm \leq \dim_\Gamma \overline{\operatorname{im} j^*}. \tag{2.27}$$

   We can now write the commutative diagram of Hilbert Γ-modules

$$\begin{array}{ccccccccc} \longrightarrow & H^s_{(2)}(\widetilde{M}) & \xrightarrow{j^*} & H^s_{(2)}(\widetilde{N}) & \xrightarrow{\delta^*} & H^{s+1}_{(2)}(\widetilde{M},\widetilde{N}) & \xrightarrow{p^*} & H^{s+1}_{(2)}(\widetilde{M}) & \longrightarrow \\ & \downarrow \wr & & & & \downarrow \wr & & \downarrow \wr & \\ \longrightarrow & H^{n-s}_{(2)}(\widetilde{M},\widetilde{N}) & \xrightarrow{\delta_*} & H^{n-s-1}_{(2)}(\widetilde{N}) & \xrightarrow{j_*} & H^{n-s-1}_{(2)}(\widetilde{M}) & \xrightarrow{p_*} & H^{n-s-1}_{(2)}(\widetilde{M},\widetilde{N}) & \longrightarrow, \end{array}$$

   where:

   - the first row is the long weakly exact L^2-cohomology sequence of Proposition 2.25;
   - the second row is the adjoint of the first row (i.e. the maps are the Hilbert adjoints of the corresponding maps in the first row); it also is weakly exact because the adjoint functor is easily shown to be (weakly) exact;
   - the vertical maps are given by the L^2-Poincaré duality of Proposition 2.25.

From commutativity and weak exactness of the rows, using additivity of the Γ-dimension (Proposition 2.2), we easily get

$$\dim_\Gamma H^*(\widetilde{N}) = \dim_\Gamma \ker \delta^* + \dim_\Gamma \overline{\operatorname{im} \delta^*} = \dim_\Gamma \overline{\operatorname{im} j^*} + \dim_\Gamma \ker p^* =$$
$$= \dim_\Gamma \overline{\operatorname{im} j^*} + \dim_\Gamma \ker p_* = \dim_\Gamma \overline{\operatorname{im} j^*} + \dim_\Gamma \overline{\operatorname{im} j_*}.$$

However, it is clear that

$$\dim_\Gamma \overline{\operatorname{im}\{j_* : H^*_{(2)}(\widetilde{N}) \to H^*_{(2)}(\widetilde{M})\}} = \dim_\Gamma \overline{\operatorname{im}\{j^* : H^*_{(2)}(\widetilde{M}) \to H^*_{(2)}(\widetilde{N})\}},$$

again by weak exactness of the adjoint functor. We have then

$$\dim_\Gamma \overline{\operatorname{im} j^*} = \frac{1}{2} \dim_\Gamma H^*_{(2)}(\widetilde{N}) = \frac{1}{2} h_\Gamma(\widetilde{B}).$$

Putting this into (2.27), we get

$$h^\pm_\Gamma \leq \frac{1}{2} h_\Gamma(\widetilde{B}).$$

Thanks to Corollary 2.18, we must have $h^+_\Gamma + h^-_\Gamma = h_\Gamma(\widetilde{B})$. By our inequality, this forces

$$h^+_\Gamma = h^-_\Gamma = \frac{1}{2} h_\Gamma(\widetilde{B}).$$

□

It remains to prove step 2, which we have posticipated. Namely, we want to identify the Γ-signature of the intersection form on $\mathcal{H}^{2k}_{(2)}(\widetilde{M}_\infty)$ with the Γ-signature of the intersection form on $\widehat{H}^{2k}_{(2)}(\widetilde{M}) = \overline{\operatorname{im}\{H^{2k}_{(2)}(\widetilde{M}, \widetilde{N}) \to H^{2k}_{(2)}(\widetilde{M})\}}$. The Atiyah-Patodi-Singer proof of Proposition 1.34 cannot be exported completely, because here the long sequence is only weakly exact and, moreover, an L^2-form which is 0 in L^2-cohomology needs not be an exact differential. The problem was circumvented by Lück and Schick in [14] by working on subspaces with better properties and arbitrary small Γ-codimension. Our treatment is directly taken by their paper.

Proposition 2.28. *The restriction map*

$$r : \mathcal{H}^*_{(2)}(\widetilde{M}_\infty) \to L^2\Omega^*(\widetilde{M})$$
$$\omega \mapsto \omega|_{\widetilde{M}}$$

induces a weak isomorphism

$$\mathcal{H}^{2k}_{(2)}(\widetilde{M}_\infty) \to \widehat{H}^*_{(2)}(\widetilde{M})$$

which transforms $\widehat{s}$ *into* s_∞*. In particular, we have*

$$\operatorname{sign}_\Gamma \widehat{s} = \operatorname{sign}_\Gamma s_\infty.$$

Proof. Denote by $[r] : \mathcal{H}^p_{(2)}(\widetilde{M}_\infty) \to H^p_{(2)}(\widetilde{M})$ the induced map, and consider the weakly exact sequence

$$\cdots \to H^p_{(2)}(\widetilde{M}, \widetilde{N}) \xrightarrow{[i]} H^p_{(2)}(\widetilde{M}) \xrightarrow{[q]} H^p_{(2)}(\widetilde{N}) \to \cdots.$$

First of all, as in Proposition 1.34, it can be shown that

$$\operatorname{im}[r] \subseteq \widehat{H}^p_{(2)}(\widetilde{M}) = \overline{\operatorname{im}[i]} = \ker[q]$$

by checking that the restriction $b(h) := h|_N$ is an exact form for any $\mathcal{H}^p_{(2)}(\widetilde{M}_\infty)$.

We want to prove that $\operatorname{im}[r]$ is dense in $\widehat{H}^p_{(2)}(\widetilde{M})$. To do so, we will find a dense subspace $K^p \subseteq \widehat{H}^p_{(2)}(\widetilde{M})$ which is contained in the image of $[r]$. In reduced L^2-cohomology, forms cohomologous to zero are not necessarily exact. Nonetheless, Lück and Schick were able to define *closed* subspaces

$$E^p_\varepsilon \subseteq L^2\Omega^p(\widetilde{N})$$

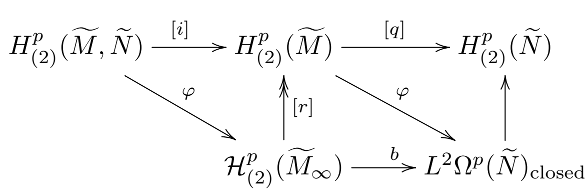
for any $\varepsilon > 0$ such that

- $E^p_\varepsilon \subseteq d(L^2\Omega^{p-1}(\widetilde{N}))$;
- E^p_ε has Γ-codimension $\leq \varepsilon$ in $\overline{d(L^2\Omega^{p-1}(\widetilde{N}))}$.

In other words, E^p_ε are closed subspaces made of exact forms and of arbitrary small Γ-codimension in $\overline{\operatorname{im} d}$. For the construction of these subspaces, we refer to [14]. Now, thanks to Proposition 2.24, we can identify $H^p_{(2)}(\widetilde{M})$ with the space of harmonic forms on M satisfying absolute boundary conditions on $\widetilde{N}$. Using this identification, we can define a continuous map

$$\varphi : H^p_{(2)}(\widetilde{M}) \to L^2\Omega^p(\widetilde{N})$$

as the restriction to the boundary of the harmonic representative. To clarify the exposition, we sketch a commutative diagram:



(here, b is just the restriction map). We define now

$$K^p_\varepsilon := \varphi^{-1}(E^p_\varepsilon).$$

By contruction, $K^p_\varepsilon \subseteq H^p_{(2)}(\widetilde{M})$ is closed and has the following properties:

- harmonic representatives of elements in K^p_ε restrict to exact forms on the boundary;
- $K^p_\varepsilon \subseteq \widehat{H}^p_{(2)}(\widetilde{M})$ has Γ-codimension $\leq \varepsilon$.

Using the first property, we show now that K^p_ε is contained in the image of $[r]$ for all ε. This fact will imply that $\mathrm{im}[r]$ is dense in $\widehat{H}^p_{(2)}(\widetilde{M})$, because $K^p := \bigcup_\varepsilon K^p_\varepsilon$ is clearly so, thanks to the second property. Let hence $[\omega] \in K^p_\varepsilon$, with ω a harmonic form on M. We want to find $h \in \mathcal{H}^p_{(2)}$ such that $[r](h) = [\omega]$, i.e. such that $h|_{\widetilde{M}}$ and ω are in the same L^2-cohomology class. By assumption, $\omega|_{\widetilde{N}} = d\alpha$ for some $\alpha \in L^2\Omega^{p-1}(\widetilde{N})$. Choose now a smooth function $\psi : (-\infty, 0] \to \mathbb{R}$ with $\psi(t) = 1$ near 0 and $\psi(t) = 0$ for $t < -1/2$. We define now $\widehat{\alpha} := \psi(t)\alpha \in L^2\Omega^p(\widetilde{N} \times [-\infty, 0])$, and

$$\widehat{\omega} := \begin{cases} d\widehat{\alpha} & \text{on } \widetilde{N} \times (-\infty, 0] \\ \omega, & \text{on } \widetilde{M} \end{cases} \in L^2\Omega^p(\widetilde{M}_\infty).$$

Integrating by parts, it can be easily shown that $\widehat{\omega}$ is orthogonal to $\delta(\Omega^{p+1}_c(\widetilde{M}_\infty))$. Thanks to the Hodge-Kodaira decomposition (1.49), then, we can write $\widehat{\omega}$ as a sum

$$\widehat{\omega} = h + \eta,$$

with $h \in \mathcal{H}^p_{(2)}(\widetilde{M}_\infty)$ and $\eta \in \overline{d(\Omega^{p-1}_c(\widetilde{M}_\infty))}$. As a consequence,

$$\omega - h|_{\widetilde{M}} = \eta|_{\widetilde{M}} \in \overline{d(L^2\Omega^{p-1}(\widetilde{M}))},$$

i.e. $[\omega] = [r](h)$.

We have thus established that $[r] : \mathcal{H}^p_{(2)}(\widetilde{M}_\infty) \to \widehat{H}^p_{(2)}(\widetilde{M})$ is a weak surjection. We now want to prove that it transforms s_∞ into $\widehat{s}$ (as in the compact case, then, injectivity of $[r]$ follows from the non-degeneracy of s_∞). We define V to be the inverse image of $E^{2k} = \bigcup_\varepsilon E^{2k}_\varepsilon$ through the restriction map

$$\mathcal{H}^{2k}_{(2)}(\widetilde{M}_\infty) \to L^2\Omega^{2k}(\widetilde{N}).$$

By definition, any element of V restricts on N to a true exact form. In particular, we can use the same arguments of Theorem 1.34 to show that

$$s_\infty(h, h') = \widehat{s}([r](h), [r](h')) \tag{2.28}$$

for any $h, h' \in V$. Now, by construction, V is a dense subspace of $\mathcal{H}^{2k}_{(2)}(\widetilde{M}_\infty)$. By continuity, (2.28) must be true on the whole space, and this concludes the proof. □

As in the compact case, we deduce from the signature formula a Novikov additivity property.

Corollary 2.29. (L^2-Novikov additivity) *Let $\widetilde{M}$ and $\widetilde{M}'$ two cocompact oriented Riemannian Γ-manifolds with boundary of dimension $4k$, with product structure near the boundary, and suppose that $\widetilde{N} \subseteq \partial\widetilde{M}$, $-\widetilde{N} \subseteq \widetilde{M}'$ are two unions of boundary components which are related by an orientation-reversing diffeomorphism. Then, the signature of the oriented manifold $\widetilde{M} \cup_{\widetilde{N}} \widetilde{M}'$ is computed by*

$$\operatorname{sign}_\Gamma(\widetilde{M} \cup_{\widetilde{N}} \widetilde{M}') = \operatorname{sign}_\Gamma \widetilde{M} + \operatorname{sign}_\Gamma \widetilde{M}'.$$

We conclude the chapter with the following bibliographical remark. In his master's thesis [26], Vaillant proves the Γ-index formula for Γ-manifolds with infinite cylindrical ends without appealing to Ramachandran's theorem for manifolds with boundary. He uses instead different methods, which, for the signature operator, lead to a Γ-signature formula for $\operatorname{sign}_\infty$. This gives an alternative way to do our steps 3-5, and it is to his work that Luck and Schick refer in [14] to conclude the proof of Theorem 2.27.

Chapter 3

Γ-rho invariants and applications

In this conclusive chapter, M will always be a closed *odd* dimensional manifold. The central object will be the so-called Γ-rho invariant associated to any Dirac operator on M and Γ-covering $\widetilde{M} \to M$. As we shall see, this is given as a difference between an Atiyah-Patodi-Singer eta invariant on M and the corresponding Γ-eta invariant of a Γ-covering $\widetilde{M} \to M$. The Γ-rho invariant is a much more stable object than the eta invariant. For example, while eta invariants associated to the odd signature operator may depend on the metric, Γ-rho invariants are metric-independent. As a consequence, they can be used in questions about the classification of differential manifolds. We shall illustrate an impressive result of Chang and Weinberger on these lines. Notice that a parallel theory can be developed for the Γ-rho invariant of the spin Dirac operator, leading to a theorem of Piazza and Schick about the space of metrics of positive scalar curvature on M.

3.1 Stability properties of Γ-rho invariants

In this section, we define Γ-rho invariants associated to a Dirac operator and a Γ-covering on a closed odd-dimensional manifold M. Then, we concentrate on the case of the odd signature operator D_{sign}. We rewrite $\rho_\Gamma(D_{\text{sign}})$ in a more workable way, and we prove that it is an oriented differential invariant of the covering $\widetilde{M} \to M$.

3.1.1 Definition of the Γ-rho invariant

Let M be a closed Riemannian odd-dimensional manifold and let D be a Dirac operator on M. Consider then a Γ-covering $\widetilde{M} \to M$ and the lifted Dirac operator $\widetilde{D}$ on $\widetilde{M}$. Whether or not M is the boundary of an even dimensional manifold, we have seen that we can define eta invariants

$$\eta(D) = \frac{1}{\sqrt{\pi}} \int_0^{+\infty} t^{-\frac{1}{2}} \operatorname{tr}(De^{-tD^2})dt,$$
$$\eta_\Gamma(\widetilde{D}) = \frac{1}{\sqrt{\pi}} \int_0^{+\infty} t^{-\frac{1}{2}} \operatorname{tr}_\Gamma(\widetilde{D}e^{-t\widetilde{D}^2})dt$$

(we drop here the italic notation $\widetilde{\mathcal{D}}$ inside the functional calculus). If we consider the covering $\widetilde{M} \to M$ as part of the structure, we can associate to D the number

$$\rho_\Gamma(D) := \eta_\Gamma(\widetilde{D}) - \eta(D) \in \mathbb{R}.$$

$\rho_\Gamma(D)$ is called the Γ-*rho* (or L^2-rho) *invariant* associated to D. It was introduced, for the signature operator, by Cheeger and Gromov in [9].

3.1.2 Metric independence for the signature operator

Suppose now that M is oriented, and let us concentrate on the case of the odd signature operator $D_{\text{sign}} : \Omega^*(M) \to \Omega^*(M)$. Thanks to Remark 1.38, we have

$$\eta(D_{\text{sign}}) = 2\eta(D^{\text{ev}}_{\text{sign}}), \qquad \eta_\Gamma(\widetilde{D}_{\text{sign}}) = 2\eta_\Gamma(\widetilde{D}^{\text{ev}}_{\text{sign}}),$$

where $D^{\text{ev}}_{\text{sign}}$ is the restriction of D_{sign} to ths space of even dimensional forms. In fact, a sharper reduction can be done, which will turn useful in the computations. For simplicity, let us fix the dimension of M to be $4k-1$, which is the case we will need in the following. Similar remarks apply to M^{2l-1} for odd l.

Proposition 3.1. *Let D_{sign} be the odd signature operator on M^{4k-1}, and let Δ_p denote the Laplacian on p-forms. Then, we have*

$$\eta(D^{\text{ev}}_{\text{sign}}) = \frac{1}{\pi} \int_0^{+\infty} t^{-\frac{1}{2}} \operatorname{tr}(d \star e^{-t\Delta_{2k}})dt = \frac{1}{\pi} \int_0^{+\infty} t^{-\frac{1}{2}} \operatorname{tr}(\star d e^{-t\Delta_{2k-1}})dt,$$
$$\eta(\widetilde{D}^{\text{ev}}_{\text{sign}}) = \frac{1}{\pi} \int_0^{+\infty} t^{-\frac{1}{2}} \operatorname{tr}(\tilde{d}\tilde{\star} e^{-t\widetilde{\Delta}_{2k}})dt = \frac{1}{\pi} \int_0^{+\infty} t^{-\frac{1}{2}} \operatorname{tr}(\tilde{\star}\tilde{d} e^{-t\widetilde{\Delta}_{2k-1}})dt.$$

Proof. Thanks to the Hodge theorem, we can decompose the space of even dimensional forms $\Omega^{\text{ev}}(M)$ as

$$\Omega^{\text{ev}}(M) = \mathcal{H}^{\text{ev}}(M) \oplus d\Omega^{\text{odd}}(M) \oplus \delta\Omega^{\text{odd}}(M).$$

Up to constants, D_{sign} acts as 0 on the first summand, as $d\star$ on the second summand and as $\star d$ on the third. Now, it is easy to verify that D_{sign}, acting as $d\star$, sends $d\Omega^{2k-1}(M)$ into itself. None of the $d\Omega^{2p-1}(M)$ (for $p \neq k$) nor $\delta\Omega^{2p-1}(M)$ (for any p) can have the same property. On these spaces, $D^{\text{ev}}_{\text{sign}}$ has a symmetric spectrum and hence gives no contribution to the eta invariant. As a consequence, $\eta(B^{\text{ev}}) = \eta(d\star)$, where $d\star$ is restricted to $d\Omega^{2k-1}(M)$ or, equivalently, to the whole $\Omega^{2k}(M)$. It follows that

$$\eta(D^{\text{ev}}_{\text{sign}}) = \frac{1}{\pi} \int_0^{+\infty} t^{-\frac{1}{2}} \operatorname{tr}(d \star e^{-td\star d\star})dt = \frac{1}{\pi} \int_0^{+\infty} t^{-\frac{1}{2}} \operatorname{tr}(d \star e^{-t\Delta_{2k}})dt.$$

From 1.38, now, it follows that $\eta(D^{\mathrm{ev}}_{\mathrm{sign}}) = \eta(D^{\mathrm{odd}}_{\mathrm{sign}})$. To get equality with $\frac{1}{\pi}\int_0^{+\infty} t^{-\frac{1}{2}} \operatorname{tr}(\star d e^{-t\Delta_{2k-1}})dt$, we repeat the same arguments as above for the space of odd-dimensional forms.

Very similar remarks apply for the Γ-eta invariant of the lifts of these operators to a Γ-covering $\widetilde{M} \to M$. It only suffices to use the Hodge decomposition 1.28 instead of the smooth one for closed manifolds. □

For brevity, we shall adopt the following notations

$$\eta(M) := \eta(D^{\mathrm{ev}}_{\mathrm{sign}}) \qquad \eta_\Gamma(\widetilde{M}) := \eta_\Gamma(\widetilde{D}^{\mathrm{ev}}_{\mathrm{sign}}) \qquad \rho_\Gamma(M) := \rho_\Gamma(D^{\mathrm{ev}}_{\mathrm{sign}}).$$

If M is the boundary of a $4k$-dimensional manifold W, provided with a Γ-covering $\widetilde{W} \to W$, the signature formula proven in the previous chapters can be thus rewritten in the form

$$\begin{aligned} \operatorname{sign} W &= \int_W L(W) - \eta(M), \\ \operatorname{sign}_\Gamma \widetilde{W} &= \int_W L(W) - \eta_\Gamma(\widetilde{M}). \end{aligned}$$

Of course the signature operator, being defined by the Hodge star operator, depends on the Riemannian metric. A striking fact is that the Γ-rho invariant $\rho_\Gamma(M)$ does not.

Theorem 3.2. (Cheeger-Gromov) $\rho_\Gamma(M)$ *is independent from the Riemannian metric on* M*.*

Proof. The proof is taken from the original paper [9]. Consider two Riemannian metrics g_0 and g_1 on M, and denote by M_i the manifold M equipped with the metric g_i. We want to show that $\rho_\Gamma(M_0) = \rho_\Gamma(M_1)$. The idea is to join g_0 and g_1 with a segment $g_u := (1-u)g_0 + ug_1$, $u \in [0,1]$, and show that the Γ-rho invariant is constant along these metrics by proving the formula

$$\frac{d}{du}\rho_\Gamma(M_u) = 0.$$

Since g_0 and g_1 are arbitrary, it is even sufficient to show the vanishing of the derivative at $u = 0$. By definition of the Γ-rho invariant, our goal is to prove that

$$\frac{d}{du}\eta_\Gamma(\widetilde{M}_u)|_{u=0} = \frac{\partial}{\partial u}\eta(M_u)|_{u=0}. \tag{3.1}$$

Let us study the second term first. We denote by $\star_u$ the Hodge star operator on M_u, and by $\dot{\star}_u$ its derivative with respect to u. Analogous notations will be used for operators involving $\star_u$ in their definition, such as δ_u and Δ_u. By Δ, we will always denote the Laplacian on $2k$-forms. Moreover, for any $t > 0$, we set $E_u(t) := e^{-t\Delta_u}$. For this function, we remark the semigroup property $E_u(t+s) = E_u(t)E_u(s)$, and the commutation with Δ_u and $d\star_u$. Thanks to Proposition 3.1, then, we can write

$$\frac{d}{du}\eta(M_u)|_{u=0} = \frac{1}{\pi}\int_0^{+\infty} t^{-\frac{1}{2}} \frac{d}{du} \operatorname{tr}(d \star_u E_u(t))|_{u=0}\, dt. \tag{3.2}$$

We want to study the term $\frac{d}{du}\,\mathrm{tr}(d\star_u E_u(t))|_{u=0}$. Observe first that

$$\frac{d}{du}\,\mathrm{tr}(d\star_u E_u(t)) = \mathrm{tr}(d\dot{\star}_u E_u(t)) + \mathrm{tr}(d\star_u \dot{E}_u(t)).$$

Moreover, we see that, for any $0<\varepsilon<t$, thanks to the semigroup property $E_u(t) = E_u(t-\varepsilon+\varepsilon) = E_u(t-\varepsilon)E_u(\varepsilon)$ we have

$$\mathrm{tr}(d\star_u \dot{E}_u(t)) = \mathrm{tr}(d\star_u \dot{E}_u(t-\varepsilon)E_u(\varepsilon)) + \mathrm{tr}(d\star_u E_u(t-\varepsilon)\dot{E}_u(\varepsilon)).$$

In particular,

$$\begin{aligned}
&\frac{d}{du}\,\mathrm{tr}(d\star_u E_u(t))|_{u=0} = \\
&\quad = \mathrm{tr}(d\dot{\star}_0 E_0(t)) + \lim_{\epsilon\to 0}\mathrm{tr}(d\star_0 \dot{E}_0(t-\varepsilon)E_0(\varepsilon)) + \mathrm{tr}(d\star_u E_0(t-\varepsilon)\dot{E}_0(\varepsilon)) = \\
&\quad = \mathrm{tr}(d\dot{\star}_0 E_0(t)) + \lim_{\epsilon\to 0}\frac{d}{du}\,\mathrm{tr}(d\star_0 E_u(t-\varepsilon)E_0(\varepsilon) - d\star_0 E_u(\varepsilon)E_0(t-\varepsilon))|_{u=0}.
\end{aligned} \tag{3.3}$$

We can now write

$$\begin{aligned}
&d\star_0 E_u(t-\varepsilon)E_0(\varepsilon) - d\star_0 E_u(\varepsilon)E_0(t-\varepsilon) = \\
&\qquad = -\int_\varepsilon^{t-\varepsilon} d\star_0 \frac{d}{ds}(E_u(t-s)E_0(s))ds = \\
&\qquad = -\int_\varepsilon^{t-\varepsilon} (-d\star_0 E_u'(t-s)E_0(s) + d\star_0 E_u(t-s)E_0'(s))ds = \\
&\qquad = -\int_\varepsilon^{t-\varepsilon} (d\star_0 \Delta_u E_u(t-s)E_0(s) - d\star_0 E_u(t-s)\Delta_0 E_0(s))ds.
\end{aligned} \tag{3.4}$$

We need to take the trace of this term. Using the semigroup property of E_0, the fact that $E_0(t)$, Δ_0 and $d\star_0$ commute, and the trace property (twice), we see that

$$\begin{aligned}
\mathrm{tr}(d\star_0 E_u(t-s)\Delta_0 E_0(s)) &= \mathrm{tr}([d\star_0 E_u(t-s)E_0(s/2)][E_0(s/2)\Delta_0]) = \\
&= \mathrm{tr}([E_0(s/2)\Delta_0][d\star_0 E_u(t-s)E_0(s/2)]) = \\
&= \mathrm{tr}([E_0(s/2)][\Delta_0 d\star_0 E_u(t-s)E_0(s/2)]) = \\
&= \mathrm{tr}([\Delta_0 d\star_0 E_u(t-s)E_0(s/2)][E_0(s/2)]) = \\
&= \mathrm{tr}(d\star_0 \Delta_0 E_u(t-s)E_0(s)).
\end{aligned}$$

Taking the trace of (3.4), hence, we get

$$\mathrm{tr}(d\star_0 E_u(t-\varepsilon)E_0(\varepsilon) - d\star_0 E_u(\varepsilon)E_0(t-\varepsilon)) = -\int_\varepsilon^{t-\varepsilon} \mathrm{tr}(d\star_0(\Delta_u - Delta_0)E_u(t-s)E_0(s))ds.$$

Recalling that we want to go on with formula (3.3), we will now study the derivative with respect to u of this last equality (calculated in $u=0$), and then let ε tend to 0. We get

$$\begin{aligned}
&\frac{d}{du}\operatorname{tr}(d\star_0 E_u(t-\varepsilon)E_0(\varepsilon) - d\star_0 E_u(\varepsilon)E_0(t-\varepsilon))|_{u=0} = \\
&\quad = -\int_\varepsilon^{t-\varepsilon} \operatorname{tr}(d\star_0 \frac{d}{du}((\Delta_u - \Delta_0)E_u(t-s))_{u=0}E_0(s))ds = \\
&\quad = -\int_\varepsilon^{t-\varepsilon} \operatorname{tr}(d\star_0 \dot{\Delta}_0 E_0(t-s)E_0(s))ds = \\
&\quad = -\int_\varepsilon^{t-\varepsilon} \operatorname{tr}(d\star_0 \dot{\Delta}_0 E_0(t))ds = \\
&\quad = -(t-2\varepsilon)\operatorname{tr}(d\star_0 \dot{\Delta}_0 E_0(t)) = \\
&\quad = -(t-2\varepsilon)\operatorname{tr}(d\star_0 (d\dot{\delta}_0)E_0(t)) = \\
&\quad = -(t-2\varepsilon)\operatorname{tr}(d\star_0 (d\dot{\star}_0 d\star_0 + d\star_0 d\dot{\star}_0)E_0(t)) = \\
&\quad = -2(t-2\varepsilon)\operatorname{tr}(E_0(t)d\star_0 d\star_0 d\dot{\star}_0) = \\
&\quad = -2(t-2\varepsilon)\operatorname{tr}(E_0(t)\Delta_0 d\dot{\star}_0) = \\
&\quad = 2(t-2\varepsilon)\operatorname{tr}(\frac{d}{dt}E_0(t)d\dot{\star}_0) = \\
&\quad = 2(t-2\varepsilon)\frac{d}{dt}\operatorname{tr}(E_0(t)d\dot{\star}_0) \xrightarrow{\varepsilon\to 0} 2t\frac{d}{dt}\operatorname{tr}(E_0(t)d\dot{\star}_0).
\end{aligned}$$

Putting this into (3.3), we find

$$\frac{d}{du}\operatorname{tr}(d\star_u E_u(t))|_{u=0} = \operatorname{tr}(d\dot{\star}_0 E_0(t)) + 2t\frac{d}{dt}\operatorname{tr}(E_0(t)d\dot{\star}_0).$$

This was the needed part of the integrand in (3.2). In particular, we have

$$\begin{aligned}
\frac{d}{du}\eta(M_u)|_{u=0} &= \frac{1}{\pi}\int_0^{+\infty} t^{-\frac{1}{2}}\left(\operatorname{tr}(d\dot{\star}_0 E_0(t)) + 2t\frac{d}{dt}\operatorname{tr}(E_0(t)d\dot{\star}_0)\right)dt = \\
&= \frac{1}{\pi}\int_0^{+\infty} t^{-\frac{1}{2}}\operatorname{tr}(d\dot{\star}_0 E_0(t))dt + \frac{1}{\pi}\int_0^{+\infty} 2t^{\frac{1}{2}}\frac{d}{dt}\operatorname{tr}(E_0(t)d\dot{\star}_0)dt.
\end{aligned}$$

Integrating by parts the second summand, it cancels out with the first summand, except for the boundary term. We get thus

$$\frac{d}{du}\eta(M_u)|_{u=0} = \frac{2}{\sqrt{\pi}}\left(\lim_{T\to+\infty}\sqrt{T}\operatorname{tr}(E_0(T)d\dot{\star}_0) - \lim_{\varepsilon\to 0}\sqrt{\varepsilon}\operatorname{tr}(E_0(\varepsilon)d\dot{\star}_0)\right). \quad (3.5)$$

The same exact arguments for the Γ-trace, for which we have presented the relevant properties in Chapter 2, yeld to

$$\frac{d}{du}\eta_\Gamma(\widetilde{M}_u)|_{u=0} = \frac{2}{\sqrt{\pi}}\left(\lim_{T\to+\infty}\sqrt{T}\operatorname{tr}_\Gamma(\widetilde{E}_0(T)\widetilde{d\dot{\star}_0}) - \lim_{\varepsilon\to 0}\sqrt{\varepsilon}\operatorname{tr}_\Gamma(\widetilde{E}_0(\varepsilon)\widetilde{d\dot{\star}_0})\right), \quad (3.6)$$

where $\widetilde{E}_0(t)$ denotes the heat operator $e^{-t\widetilde{\Delta}_0}$ on $\widetilde{M}_0$ (the Laplacian acting on $2k$-forms).

We now want to compare (3.5) and (3.6), showing that their difference is 0. Namely, we will prove that

$$\lim_{T\to+\infty} \sqrt{T}\,\mathrm{tr}(E_0(T)d\dot{\star}_0) = 0 = \lim_{T\to+\infty} \sqrt{T}\,\mathrm{tr}(\widetilde{E}_0(T)\tilde{d}\tilde{\dot{\star}}_0), \tag{3.7}$$

and

$$\lim_{\varepsilon\to 0} \sqrt{\varepsilon}\,\mathrm{tr}(E_0(\varepsilon)d\dot{\star}_0) = \lim_{\varepsilon\to 0} \sqrt{\varepsilon}\,\mathrm{tr}(\widetilde{E}_0(\varepsilon)\tilde{d}\tilde{\dot{\star}}_0). \tag{3.8}$$

Now, (3.8) holds thanks to 2.10. To prove (3.7), notice that

$$\mathrm{tr}(E_0(T)d\dot{\star}_0) = \mathrm{tr}(E_0(T)d\star_0\star_0^{-1}\dot{\star}_0) \leq \|\star_0^{-1}\dot{\star}_0\|\,\mathrm{tr}(|E_0(T)d\star_0|),$$

so, for the first equality, it suffices to estimate the limit as $T\to+\infty$ of $\sqrt{T}\,\mathrm{tr}(|d\star_0 E_0(T)|)$. Now, we have

$$\sqrt{T}\,\mathrm{tr}(|d\star_0 E_0(T)|) \leq \mathrm{tr}(D^{\mathrm{ev}}_{\mathrm{sign}}e^{-T(D^{\mathrm{ev}}_{\mathrm{sign}})^2}) = \sqrt{T}\frac{1}{2}\,\mathrm{tr}(D_{\mathrm{sign}}e^{-TD^2_{\mathrm{sign}}}),$$

where D_{sign} is the signature operator for the metric g_0. This function is integrable at $+\infty$ thanks to the computations in Corollary 1.41. A fortiori, it converges to 0. An analogous argument leads to

$$\sqrt{T}\,\mathrm{tr}_\Gamma(\widetilde{E}_0(T)\tilde{d}\tilde{\dot{\star}}_0) \xrightarrow{T\to+\infty} 0$$

(see Proposition 2.13). □

3.2 An application of Γ-rho invariants in geometric topology

Let M^{4k-1} be a closed oriented manifold of dimension $4k-1$. It may seem that, without any other structure on M, the only Γ-rho invariant that we can naturally associate to M and D_{sign} is $\rho_{\pi_1(M)}(M)$, the rho invariant of the fundamental cover. In their paper [8], Chang and Weinberger define yet another natural invariant by L^2-cohomological means, which is shown to be a Γ-rho invariant associated to a different covering. The analysis we have studied so far is used to recognise this identification, and to prove the independence on the metric without appealing to a de Rham theorem for L^2-cohomology.

Chang and Weinberger defined the above invariant in order to prove an impressing result in manifold classification theory. Their proof is carried in the sophisticated framework of surgery theory, so we will not try to present it in a complete and rigorous way. Taking some results for granted, however, it is still possible to follow and appreciate the main steps of their very clean argument.

3.2.1 The Chang-Weinberger Γ-rho invariant

By Thom's classical results in cobordism theory, for any closed oriented M^{4k-1}, we can find an oriented $4k$-dimensional manifold W such that

$$\partial W = rM,$$

i.e. such that its oriented boundary is the disjoint union of r copies of M. By the work of Hausmann (cited in [8]), moreover, we can choose W such that the induced map $\pi_1(M) \to \pi_1(W)$ is an injection. Fix any Riemannian metric on W with product structure near the boundary, and lift it to the fundamental cover $\widetilde{W}$ of W. Define now

$$\rho_{\mathrm{cw}}(M) := \frac{1}{r}(\operatorname{sign} W - \operatorname{sign}_{\pi_1(W)} \widetilde{W}).$$

We call $\rho_{\mathrm{cw}}(M)$ the *Chang-Weinberger* Γ*-rho invariant* of M. A priori, $\rho_{\mathrm{cw}}(M)$ may depend on the choice of W as a Riemannian manifold. In fact, it does not, as we are going to show.

First of all, we prove that $\rho_{\mathrm{cw}}(M)$ is independent on the Riemannian metric. This descends from an analogous property which is true for the whole L^2-cohomology, but we can also deduce it from Theorem 3.2. In fact, $\rho_{\mathrm{cw}}(M)$ is a Γ-rho invariant associated to the signature operator. This is shown thanks to our signature theorems 1.37 and 2.27. We have

$$\begin{aligned} \operatorname{sign} W &= \int_W L(W) - r\eta(M), \\ \operatorname{sign}_{\pi_1(W)} \widetilde{W} &= \int_W L(W) - r\eta_{\pi_1(W)}(\widetilde{M}_W), \end{aligned}$$

and hence

$$\rho_{\mathrm{cw}}(M) = \rho_{\pi_1(W)}(M).$$

Then, we show that $\rho_{\mathrm{cw}}(M)$ is independent on the choice of W as a differential manifold.

Proposition 3.3. $\rho_{\mathrm{cw}}(M)$ *does not depend on the differential manifold* W*, as long as the injection* $\pi_1(M) \hookrightarrow \pi_1(W)$ *is satisfied.*

Proof. Let W' another manifold such that $\partial W' = r'M$ and $\pi_1(M) \hookrightarrow \pi_1(W')$. For simplicity, we will suppose that $r' = r$: for the general case, it will suffice to pass to a common multiple of the two numbers. We put on W' a Riemannian metric coinciding near the boundary with that of W. Consider the closed oriented Riemannian manifold $X : W \cup_{rM} (-W')$, its universal cover $\widetilde{X}$ and the induced covers $\widetilde{W}_X \to W$ and $\widetilde{W}'_X \to W'$. By Atiyah's L^2-signature theorem and Novikov additivities, we have

$$0 = \operatorname{sign} X - \operatorname{sign}_{\pi_1(X)} \widetilde{X} = (\operatorname{sign} W - \operatorname{sign}_{\pi_1(X)} \widetilde{W}_X) - (\operatorname{sign} W' - \operatorname{sign}_{\pi_1(X)} \widetilde{W}'_X).$$

By Van Kampen's theorem, however, both $\pi_1(W)$ and $\pi_1(W')$ injects into $\pi_1(X)$, because $\pi_1(M)$ injects into them. As a consequence, we have

$$\operatorname{sign}_{\pi_1(X)} \widetilde{W}_X = \operatorname{sign}_{\pi_1(W)} \widetilde{W}$$

and the same for W', thanks to the Γ-induction property for Hilbert Γ-modules (see [9], equation (2.3), or [15], Lemma 1.24). In particular,

$$\begin{aligned} &\frac{1}{r}(\operatorname{sign} W - \operatorname{sign}_{\pi_1(W)} \widetilde{W}) = \frac{1}{r}(\operatorname{sign} W - \operatorname{sign}_{\pi_1(X)} \widetilde{W}_X) = \\ &= \frac{1}{r}(\operatorname{sign} W - \operatorname{sign}_{\pi_1(X)} \widetilde{W}'_X) = \frac{1}{r}(\operatorname{sign} W - \operatorname{sign}_{\pi_1(W')} \widetilde{W}'). \end{aligned}$$

□

Corollary 3.4. $\rho_{\mathrm{cw}}(M)$ *is an oriented diffeomorphism invariant of* M.

3.2.2 The surgery structure set

We introduce now the *surgery structure set* of M^{4k-1}, denoted by $\mathcal{S}(M)$. For techincal reasons, we will suppose that $k \geq 2$. First of all, we define a smooth cobordism W between two manifolds X_1, X_2 to be an *h-cobordism* if the injections $X_1 \hookrightarrow W$, $X_2 \hookrightarrow X$ are homotopy equivalences. Then, we define $\mathcal{S}(M)$ as the set of smooth homotopy equivalences $f : X \to M$ (where X is another closed manifold) modulo the following equivalence relation. We say that two homotopy equivalences $f_0 : X_0 \to M$, $f_1 : X_1 \to M$ are equivalent if there exists an h-cobordism W between X_0 and X_1 and a smooth homotopy equivalence

$$F : W \to M \times [0,1]$$

which restricts on the boundary to $f_0 : X_0 \to M \times \{0\}$ and $f_1 : X_1 \to M \times \{1\}$. In $\mathcal{S}(M)$ there is of course a preferred base point , which is the class of the identity map $M \to M$.

The surgery structure set is the central object in surgery theory, which is a series of techniques aimed towards the classification of manifolds. It fits in a sequence of maps called the *surgery exact sequence*. We only need one map in this sequence. This is in fact an action of the *L-group* $L_{4k}(\mathbb{Z}[\pi])$ on the structure set ($\pi := \pi_1(M)$), but we will see it as map by only studying the action on the base point $\mathrm{id} : M \to M$. We will be very naive in the description of $L_{4k}(\pi)$. Rigorous definitions and proofs can be found in [19]. We limit ourselves to say that $L_{4k}(R)$ is defined algebraically as the set of quadratic forms (i.e. symmetric bilinear forms with some additional structure) over finitely generated free modules over the ring R, provided with an equivalence relation (the one that ignores hyperbolic forms) which makes it into a group with respect to the direct sum. In our case, R will be the group ring $\mathbb{Z}[\pi]$. The map

$$\alpha : L_{4k}(\mathbb{Z}[\pi]) \to \mathcal{S}(M)$$

is then defined thanks to a deep result in surgery theory called the *Wall realisation theorem*. This theorem only works in dimension ≥ 5, hence, from now on, we have to impose the restriction $k \geq 2$. By the realisation theorem, every form $\lambda \in L_{4k}(\mathbb{Z}[\pi])$ arises as the $\mathbb{Z}[\pi]$-intersection form of a degree one map $W \to M \times [0,1]$, where W is an h-cobordism between M and another manifold M', which restricts to the identity $\mathrm{id} : M \to M \times \{0\}$ on M and to a homotopy equivalence $M' \to M \times \{1\}$. The map α is then given as

$$[W \to M \times [0,1]] \mapsto [M' \to M].$$

Notice that $L(\cdot)$ is a covariant functor from Rings to Abelian Groups: given a ring homomorphism $\varphi : R \to R'$ and a quadratic form on an R-module K, we can naturally push it to a quadratic form on the R'-module $K \times_R R'$. A signature homomorphism is then defined by the composition

$$\operatorname{sign} : L_{4k}(\mathbb{Z}[\pi]) \to L_{4k}(\mathbb{Z}) \to L_{4k}(\mathbb{R}) \to \mathbb{Z},$$

where the last map is the ordinary signature on finite-dimensional vector spaces. There also is a homomorphism

$$\operatorname{sign}_\pi : L_{4k}(\mathbb{Z}[\pi]) \to \mathbb{R}.$$

For an algebraic definition, see [14]. Thanks to the realisation theorem, however, it turns out that we can see both maps geometrically, i.e. as the signature of the cobordism W^{4k} and the π-signature of its universal cover $\widetilde{W}^{4k}$ respectively. We will concentrate most on this geometric point of view, but the algebraic one is also needed at some point.

3.2.3 Chang-Weinberger theorem

We are ready to state the desired result, and to sketch a proof for it.

Theorem 3.5. (Chang, Weinberger) *Let M^{4k-1} a $(4k-1)$-dimensional closed smooth manifold ($k \geq 2$) such that $\pi_1(M)$ contains torsion. Then, there exists an infinite family $\{M_i\}_{i\in\mathbb{N}}$ of different smooth manifolds M_i which are hompotopy equivalent to M but not diffeomorphic to it.*

Proof. Let $\pi := \pi_1(M)$. We can consider the diagram

$$\begin{array}{ccc} L_{4k}(\mathbb{Z}[\pi]) & \xrightarrow{\alpha} & \mathcal{S}(M)\ , \\ & \overset{s_\pi}{\searrow} & \downarrow \tau \\ & & \mathbb{R} \end{array}$$

where s_π is the homomorphism $\operatorname{sign} - \operatorname{sign}_\pi$ and the vertical map is given by

$$\tau([M' \to M]) := \rho_{\mathrm{cw}}(M) - \rho_{\mathrm{cw}}(M').$$

The diagram commutes. In fact, suppose that an element $\lambda \in L_{4k}(\mathbb{Z}[\pi])$ is represented by a map $Y \to M \times [0,1]$, Y being an h-cobordism between M and M'. Construct a manifold W whose boundary is rM and such that π injects into $\Pi := \pi_1(W)$, so that

$$\rho_{\mathrm{cw}}(M) = \frac{1}{r}(\operatorname{sign} W - \operatorname{sign}_\Pi \widetilde{W}).$$

We can attach to W r copies of Y, obtaining an oriented manifold $W' := W \cup_{rM} (-Y)$. We have then $\partial W' = rM'$, and $\pi = \pi_1'(M) = \pi_1(M')$ injects into $\Pi = \pi_1(W')$, so that

$$\rho_{\mathrm{cw}}(M') = \frac{1}{r}(\operatorname{sign} W' - \operatorname{sign}_\Pi \widetilde{W}').$$

Using Novikov additivity on W', we find now

$$\begin{aligned}\tau\circ\alpha(\lambda) &= \tau[M'\to M] = \rho_{\rm cw}(M)-\rho_{\rm cw}(M') = \\ &= \frac{1}{r}(\operatorname{sign} W - \operatorname{sign}_\Pi \widetilde{W} - \operatorname{sign} W' + \operatorname{sign}_\Pi \widetilde{W}') = \\ &= \frac{1}{r}(r\operatorname{sign}_Y - r\operatorname{sign}_\Pi \widetilde{Y}'_W) = \\ &= \operatorname{sign} Y - \operatorname{sign}_\Pi \widetilde{Y}'_W,\end{aligned}$$

where $\widetilde{Y}'_W$ is the Π-covering on Y induced by $\widetilde{W}'\to W'$. However, we see that $\pi_1(Y)=\pi$ injects into Π. Hence, by the Γ-induction property (already cited in the proof of Proposition 3.3), we have $\operatorname{sign}_\Pi \widetilde{Y}'_W = \operatorname{sign}_\pi(\widetilde{Y})$. It follows that

$$\tau\circ\alpha(\lambda) = \operatorname{sign} Y - \operatorname{sign}_\pi \widetilde{Y} = (\operatorname{sign}-\operatorname{sign}_\pi)(\lambda) = s_\pi(\lambda).$$

Let us show that s_π has infinite image. Since it is a homomorphism, it suffices to show that it is not the trivial one. We use here the hypothesis that π is not torsion free. This means that it exists an injection

$$i:\mathbb{Z}_n\hookrightarrow\pi$$

for some $n\geq 1$. The injection i induces a commutative diagram

$$\begin{array}{ccc} L_{4k}(\mathbb{Z}[\mathbb{Z}_n]) & \xrightarrow{i_*} & L_{4k}(\mathbb{Z}[\pi]) \\ & \searrow^{s_{\mathbb{Z}_n}} & \downarrow s_\pi \\ & & \mathbb{R} \end{array},$$

where $s_{\mathbb{Z}_n}=\operatorname{sign}-\operatorname{sign}_{\mathbb{Z}_n}$. Hence, we have reduced the problem to show that $s_{\mathbb{Z}_n}$ is not the trivial homomorphism. This is shown algebraically in [8], to which we refer for the explicit example of a suitable quadratic form.

Hence, s_π has infinite image. Let $\{\lambda_i\}_{i\in\mathbb{N}}\subseteq L_{4k}(\mathbb{Z}[\pi])$ a countable family of elements such that $s_\pi(\lambda_i)\neq 0$ for all i and $s_\pi(\lambda_i)\neq s_\pi(\lambda_j)$ for $i\neq j$. The image of λ_i through $\alpha: L_{4k}(\mathbb{Z}[\pi])\to\mathcal{S}(M)$ is represented by a (degree one) homotopy equivalence $M_i\to M$. From the commutativity of the first triangular diagram, we see that

$$\rho_{\rm cw}(M)-\rho_{\rm cw}(M_i)=\alpha(\lambda_i)\neq 0 \quad \text{for all } i.$$

In particolar, since $\rho_{\rm cw}$ is an oriented diffeomorphism invariant, the manifolds M_i are all non diffeomorphic to M. Moreover, we have

$$\rho_{\rm cw}(M_i)-\rho_{\rm cw}(M_j)=\alpha(\lambda_j)-\alpha(\lambda_i)\neq 0 \quad \text{for } i\neq j,$$

so these manifolds are also mutually non diffeomorphic. □

Remark 3.6. *We cite here the existence of a parallel theory in which differentiable manifolds are replaced by spin manifolds M which admit metrics of* *positive scalar curvature, and the signature operator by the spin Dirac operator. The associated* Γ*-rho invariant may be then used to distuinguish the "*Γ*-bordism" class of two metrics of positive scalar curvature on* M *(here* Γ *is the fundamental group of* M*, and the "*Γ*-bordism" is an equivalence relation on the set* $\mathcal{M}^+(M)$ *of metics of positive scalar curvature on* M*). Piazza and Schick proved in [17] the following theorem, that may be seen as a spin-version of the result of Chang-Weinberger.*

Theorem 3.7. (Piazza, Schick) *Let* M^{4k-1} *a* $(4k-1)$*-dimensional spin manifold* $(k \geq 2)$ *admitting metrics of positive scalar curvature and such that* $\Gamma := \pi_1(M)$ *contains torsion. Then, there exists infinitely many different* Γ*-bordism classes of positive scalar curvatures on* M*.*

Bibliography

[1] P. Antonini. The Atiyah–Patodi–Singer signature formula for measured foliations. *Journal für die reine und angewandte Mathematik (Crelles Journal)*, 2013.

[2] M.F. Atiyah. Elliptic operators, discrete groups and von Neumann algebras. In *Colloque "Analyse et Topologie" en l'honneur de Henri Cartan*, number 32–33 in Astérisque, pages 43–72. Société Mathématique de France, Paris, 1976.

[3] M.F. Atiyah, V.K. Patodi, and I. Singer. Spectral asymmetry and Riemannian geometry. I. *Mathematical Proceedings of the Cambridge Philosophical Society*, 77:43–69, 1975.

[4] M.F. Atiyah, V.K. Patodi, and I. Singer. Spectral asymmetry and Riemannian geometry. II. *Mathematical Proceedings of the Cambridge Philosophical Society*, 78:405–432, 1975.

[5] M.F. Atiyah, V.K. Patodi, and I. Singer. Spectral asymmetry and Riemannian geometry. III. *Mathematical Proceedings of the Cambridge Philosophical Society*, 79:71–99, 1976.

[6] J.-M. Bismut and D.S. Freed. The analysis of elliptic families. ii. dirac operators, eta invariants, and the holonomy theorem. *Communications in Mathematical Physics*, 107(1):103–163, 1986.

[7] B. Booß-Bavnbek and K.P. Wojciechhowski. *Elliptic Boundary Problems for Dirac Operators*. Mathematics: Theory & Applications. Birkhäuser, 1993.

[8] S. Chang and S. Weinberger. On invariants of Hirzebruch and Cheeger-Gromov. *Geometric Topology*, 7:311–319, 1985.

[9] J. Cheeger and M. Gromov. Bounds on the von Neumann dimension of L^2-cohomology and the Gauss-Bonnet theorem for open manifolds. *Journal of Differential Geometry*, 21(1):1–34, 1985.

[10] J.B. Conway. *A Course in Functional Analysis*. Graduate Texts in Mathematics. Springer, 1990.

[11] G. de Rham. *Differentiable manifolds: forms, currents, harmonic forms.* Grundlehren der mathematischen Wissenschaften. Springer-Verlag, 1984.

[12] J.A. Dieudonné. *Treatise on Analysis, Vol. 7.* Academic Press, 1988.

[13] H.B. Lawson and M.L. Michelsohn. *Spin Geometry.* Princeton mathematical series. Princeton University Press, 1989.

[14] W. Lück and T. Schick. Various L^2-signature formulas and a topological signature theorem. In *High-Dimensional Manifold Topology*, chapter 15, pages 362–399. 2002.

[15] W. Lück. *L2-Invariants: Theory and Applications to Geometry and K-Theory.* A Series of Modern Surveys in Mathematics Series. Springer, 2002.

[16] R. Melrose. *The Atiyah-Patodi-Singer Index Theorem.* Research Notes in Mathematics. Taylor & Francis, 1993.

[17] P. Piazza and T. Schick. Groups with torsion, bordism and rho invariants. *Pacific Journal of Mathematics*, 232(2):355–378, 2007.

[18] M. Ramachandran. Von Neumann index theorems for manifolds with boundary. *Journal of Differential Geometry*, 38(2):315–349, 1993.

[19] A. Ranicki. *Algebraic and Geometric Surgery.* Oxford Mathematical Monographs. Oxford University Press, Oxford, 2002.

[20] M. Reed and B. Simon. *Methods of Modern Mathematical Physics, I: Functional Analysis.* Elsevier Science, 1981.

[21] J. Roe. *Elliptic Operators, Topology, and Asymptotic Methods, Second Edition.* Chapman & Hall/CRC Research Notes in Mathematics Series. Taylor & Francis, 1999.

[22] T. Schick. *Analysis on ∂-manifolds of bounded geometry, Hodge-de Rham isomorphism and L^2-index theorem.* PhD thesis, Johannes Gutenberg-Universität Mainz, 1996.

[23] T. Schick. The topology of positive scalar curvature. 2014. arXiv:1405.4220.

[24] M. Shubin. *Von Neumann Algebras and L2-Techniques in Geometry and Topology.* Online notes: http://www.mccme.ru/ium/f02/vNa.html.

[25] S. Stolz. Manifolds of positive scalar curvature. In *Topology of High-Dimensional Manifolds*, number 1,2, pages 665–709. Trieste, 2001.

[26] B. Vaillant. Index Theory for Coverings. 1997. arXiv:0806.4043.